REVISITING SHAPLEY AND SHUBIK (1971) VIA NASH (1953): A REFINEMENT OF THE CORE IN THE ASSIGNMENT GAME

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ABSTRACT. We refine the core of the assignment game to incorporate the outside option principle and the dynamic implications of rematching, in the spirit of the Nash (1953) Program. The refinement excludes a stable payoff if an objecting player is better off under every payoff-preserving rematching that isolates the objector in a single-match subgame where the outside option principle determines payoffs. It captures bargaining leverage and rematching chains that the blocking-pair formulation of the core overlooks. We construct two operators, each clearing one side's objections through the smallest increases in that side's payoffs consistent with stability, without creating objections on the other side. Applied in succession, the operators define a tâtonnement process that reaches the refined core from any stable payoff in a uniformly bounded number of steps. The refined core is the set of fixed points of their composition and forms a complete lattice, though not a sublattice of the core. The operators deliver explicit formulae for its meet, join, and side-optimal payoffs. A demand game with bargaining in single-match subgames provides non-cooperative foundations: as frictions vanish, the refined core coincides with the set of limit payoffs of refined subgame perfect equilibria.

Keywords: assignment game, core, stable payoff, bargaining, objections, rematching chains, non-cooperative implementation, outside option principle, refinements. *JEL Classification:* C78.

1. INTRODUCTION

Recall the assignment game of Shapley and Shubik (1971). The set of players N is partitioned into two "sides" I and J (buyers and sellers, men and women, and so on), and each player can match with at most one player on the opposite side. A *match* (i, j) between $i \in I$ and $j \in J$ creates a value $a_{ij} \geq 0$, which i and j share. A *matching* specifies a set of disjoint pairs in $I \times J$ that get matched, with the possibility that some players remain unmatched. A payoff profile $v = (v_k)_{k \in N}$ is *stable* if it assigns non-negative payoffs to all players, it is not vulnerable to "blocking" by pairs ($v_i + v_j \geq a_{ij}$ for all $(i, j) \in I \times J$), and it is *feasible* under some matching. Feasibility requires that matched pairs (i, j) share the value a_{ij} ($v_i + v_j = a_{ij}$), while unmatched players k get zero payoff ($v_k = 0$). Let $\mathcal{S}$ denote the set of stable payoff profiles.

A matching is *efficient* if it maximizes the sum of matched pairs' values. Every stable payoff profile is feasible under all efficient matchings (and no others). Moreover, $\mathcal{S}$ constitutes the *core*

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of a cooperative game with transferable utility in which the worth of every coalition equals the maximum total value over all matchings among members of that coalition. For simplicity, we assume that match values are generic, which yields a unique efficient matching η and positive match values a_{ij}. Let U denote the set of players not matched under η. Given that every match creates positive value, all players in U must be on the same side of the market.

Shapley and Shubik showed that $\mathcal{S}$ is a (nonempty) complete lattice with respect to the partial order $\geq_I$ on stable payoffs derived from Pareto dominance for players in I. Thus, there is an I-optimal stable payoff $\bar{v}^I$, which is most favorable across $\mathcal{S}$ to players in I (and least favorable to J), and a J-optimal stable payoff $\bar{v}^J$ at the other extreme.

$\mathcal{S}$ is conventionally viewed as "too large." One can see this starkly when I and J are singletons: $I = \{1\}, J = \{2\}$. In this case, $\mathcal{S}$ consists of every division of a_{12}. It is natural to look for an equitable selection. In this example, the equal split $v_1 = v_2 = a_{12}/2$ is an uncontroversial choice.

Our goal is to investigate refinements of $\mathcal{S}$ in the spirit of the Nash (1953) Program, where an idealized (or "cooperative") solution is also supported by a non-cooperative mechanism whose Nash equilibria—possibly refined as in Nash's demand game—yield outcomes that mirror the idealized solution. These dual perspectives reinforce each other. Our motivation echoes Shapley and Shubik's own critique of the core. As they note in their abstract, "a critique of the core solution reveals an insensitivity to some of the bargaining possibilities inherent in the situation, and indicates that further analysis would be desirable using other game-theoretic solution concepts."

A basic desideratum for our idealized solution is equity: if $i \in I$ matches with $j \in J$, then $v_i = v_j = a_{ij}/2$. Equal division is compelling when I and J are singletons. However, unconstrained equity ignores the outside options provided by alternative matches when there are competing players on either side of the market—it is incompatible with stability.

In a seminal contribution integrating stability and equity desiderata, Rochford (1984) proposed a refinement of $\mathcal{S}$ based on a natural extension of the *Nash (1950) bargaining solution* to the assignment game. For a candidate solution v, player k's *disagreement payoff* (or "outside option") is given by the best available alternative match:

$$o_k(v) = \max\{\max(a_{kl} - v_l, 0) \mid l \in L \setminus \{\eta(k)\}\},$$

where L denotes the set of players on the opposite side from k (and $\max \emptyset = 0$). Rochford's refinement $\mathcal{S}^R$ requires that for any solution v, every pair (i, j) matched under η share their gains a_{ij} according to Nash bargaining relative to the outside options $(o_i(v), o_j(v))$. Formally,

$$\mathcal{S}^R = \left\{ v \in \mathcal{S} \mid \forall k \in N \setminus U : v_k = o_k(v) + \frac{a_{k\eta(k)} - o_k(v) - o_{\eta(k)}(v)}{2} \right\}.$$

Results due to Rochford (1984), Roth and Sotomayor (1988) and Driessen (1998) establish that $\mathcal{S}^R$ is a complete lattice and coincides with the kernel of the associated cooperative game.[1]

[1]The solution has regained attention more recently: it was re-derived by Cook and Yamagishi (1992) and Kleinberg and Tardos (2008), with follow-up papers including Azar et al. (2009), Azar et al. (2010), Bateni et al. (2010), and Bayati et al. (2015).

Single-match assignment games—in which either $|I| = 1$ or $|J| = 1$—are a building block and test case for our analysis. In such a game with $I = \{1\}, J = \{2, \ldots, n\}$ and $a_{12} > \ldots > a_{1n}$, the set of stable payoffs is

$$\mathcal{S} = \{v \mid v_1 \in [a_{13}, a_{12}], v_2 = a_{12} - v_1, v_k = 0 \text{ for } k \geq 3\}$$

(for $n = 2$, set $a_{13} = 0$). In this class of games, Rochford's solution is uniquely given by $v_1 = (a_{12} + a_{13})/2, v_2 = (a_{12} - a_{13})/2$, and $v_k = 0$ for $k \geq 3$.

We are not aware of a non-cooperative implementation of Rochford's solution that would provide the mutually reinforcing foundations advocated by Nash (1953). In fact, the most commonly used bargaining protocols in single-match assignment games are incompatible with Rochford's solution, and instead support equitable divisions to the maximum extent permitted by stability constraints. This is the case in several standard bargaining models—random matching (Rubinstein and Wolinsky 1990), strategic matching (Manea 2018), and the random proposer protocol (Elliott and Nava 2019). In all these models, Markov perfect equilibria (MPEs) converge, as players become patient, to the prediction of the *outside option principle (OOP)*: players 1 and 2 match, and the payoffs ω are given by[2]

$$\omega_1 = \max\left(\frac{a_{12}}{2}, a_{13}\right), \ \omega_2 = a_{12} - \omega_1, \omega_k = 0 \text{ for } k \geq 3. \tag{1}$$

Abreu and Manea (2026) provide further support for the OOP under subgame perfect equilibrium (SPE) refinements weaker than Markov behavior. Binmore, Shaked and Sutton (1989) also report experimental evidence for this form of the OOP, which they call "deal-me-out." In their experiment, deal-me-out predicts subjects' agreements far better than the "split-the-difference" rule embodied in Rochford's solution. Their discussion emphasizes that an outside option affects a player's payoff only when the threat to exercise it is credible.

Thus, in single-match assignment problems, the OOP yields a sharp refinement of the set of stable payoffs that abides by the precepts of the Nash Program. It is the unique such solution across the three bargaining protocols above. The OOP embodies the idea that outside options justify a favorable departure from equal split only if they are binding. This leads to an OOP-consistent refinement of $\mathcal{S}$, in the spirit of Rochford's fixed-point approach:[3]

$$\mathcal{S}^* = \left\{v \in \mathcal{S} \mid \forall k \in N \setminus U : v_k > \frac{a_{k\eta(k)}}{2} \implies v_k = o_k(v)\right\}.$$

The principle that favorable deviations from equal split should be justified by binding outside options is also expressed in the bargaining models of Elliott and Nava (2019), Ho and Lee (2019)

[2]The OOP was first developed by Binmore, Shaked and Sutton (1989) in a model in which the outside option was given exogenously and not in the form of another potential strategic partner.

[3]Moldovanu (1993) introduced a general class of solutions v for which the division within each efficient match (i, j) depends only on the match value a_{ij} and the outside options $(o_i(v), o_j(v))$. While his analysis does not privilege any particular solution, $\mathcal{S}^*$ is a special case (as is $\mathcal{S}^R$), and his results imply that $(\mathcal{S}^*, \geq_I)$ is a complete lattice.

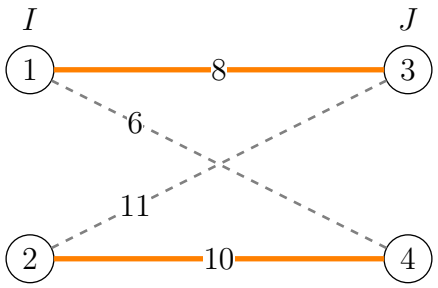

FIGURE 1. A 2×2 assignment game

and Talamas (2026). We are not aware, however, of any prior systematic formulation of this principle for the assignment game.[4]

The refinement $\mathcal{S}^*$—although suitable for single-match problems—is not fully satisfactory in slightly more complicated examples. We illustrate some of the relevant considerations in the example from Figure 1, where $I = \{1, 2\}, J = \{3, 4\}$ and $a_{13} = 8, a_{24} = 10, a_{14} = 6, a_{23} = 11$. In this example the (unique) efficient matching pairs 1 with 3, and 2 with 4.

Consider the stable payoff profile $(v_1, v_2, v_3, v_4) = (2, 5, 6, 5) \in \mathcal{S}^*$. Player 1 could "refuse to play ball" and point out to players 2 and 3 that they are able to obtain their proposed payoffs by matching with each other ($v_2 + v_3 = a_{23}$). Once the match $(2, 3)$ forms, players 1 and 4 need to reach an agreement, and an equal division seems reasonable. This would yield player 1 a higher payoff of $a_{14}/2 = 3 > v_1$. In our conservative approach to valid objections, player 1 does not have the power to specify the reassignment. Perhaps players 2 and 4 will maintain their match, isolating player 1 with player 3 in a two-player subgame. But in this case too, player 1 obtains a higher payoff, $a_{13}/2 = 4$. Player 1 is thus better off in either scenario, so her objection is legitimate.

Now consider $w = (3.5, 6.5, 4.5, 3.5) \in \mathcal{S}^*$. Player 1 might try the same argument, but here it is less compelling: her objection may lead players 2 and 3 to rematch and receive their proposed payoffs $(6.5, 4.5)$, with player 1 then matching with 4. An equal split in this subgame would now leave player 1 with a payoff of $a_{14}/2 = 3 < 3.5 = w_1$. We explain below that a more permissive approach that allows an objecting player to specify a rematching would, in general, be incompatible with stability, and consequently with existence of the proposed refinement.

The example illustrates our approach to refining the set of stable payoffs (and $\mathcal{S}^*$). When does a player $i \in I$ matched under η have a legitimate objection to a proposed stable payoff profile v? One obviously feasible rematching isolates player i with her efficient match $\eta(i)$ and the set U of players not matched under η, and implements η for the other players. If i and $\eta(i)$ individually objected to the proposed v and insisted on being isolated in this single-match game, then v could withstand their objections only by assigning the OOP payoffs to both players. Indeed, since $(i, \eta(i))$ is the

[4]Equilibria in Elliott and Nava's (2019) random-proposer bargaining model belong to the core only in a special class of assignment games. Ho and Lee's (2019) model of bargaining with threat of replacement does not nest the assignment game because it assumes that a hospital acting as an insurer's outside option maintains its other agreements when it replaces a network hospital. Talamas (2026) focuses on coalitional bargaining and implicitly assumes a steady-state market.

most valuable match among players in $\{i, \eta(i)\} \cup U$, the OOP payoffs ω in this single-match game satisfy $\omega_i + \omega_{\eta(i)} = a_{i\eta(i)}$. Then $v_i \geq \omega_i$, $v_{\eta(i)} \geq \omega_{\eta(i)}$, and $v_i + v_{\eta(i)} = a_{i\eta(i)} = \omega_i + \omega_{\eta(i)}$ together yield $v_i = \omega_i$ and $v_{\eta(i)} = \omega_{\eta(i)}$. When gains in all η matches are divided in this fashion, we obtain the "shifted Rubinstein payoffs," which Elliott and Nava (2019) show to be the unique candidate for efficient equilibria in their model. These payoffs reflect only outside options provided by players in U, and are in general incompatible with stability. When $U = \emptyset$, they reduce to equal split in every η match. In the 2×2 example above, this leads to unstable payoffs: $v_1 = v_3 = 4$ and $v_2 = v_4 = 5$. Allowing each matched player to object by proposing exactly how other players should match to obtain their proposed payoffs is therefore too permissive.

We take a conservative approach to objections: an objection by a player $i \in I \setminus U$ to a candidate payoff $v \in \mathcal{S}$ leads to a rematching of all efficiently matched players except i and a single previously matched j. The rematched players retain their payoffs under v, and i's payoff is given by her OOP payoff in the single-match game with the players in $\{j\} \cup U$. This approach privileges the original v, avoiding extreme deterrents based on distant payoff adjustments; i's objection may affect the payoff of at most one player $j \in J$ matched under η. We consider i to have a legitimate objection if i's payoff is strictly higher in all such rematchings. Our main refinement $\mathcal{S}^{**}$ excludes only stable payoffs v for which there is some player in $N \setminus U$ who has a legitimate objection.

Figure 2 compares the core, Rochford's solution, our two refinements, the Shapley (1953) value, and the nucleolus (Schmeidler 1969) for the assignment game in Figure 1. The nucleolus belongs to the kernel $\mathcal{S}^R$, whereas the Shapley value lies outside the core.

We begin our analysis of $\mathcal{S}^{**}$ by constructing a tâtonnement process that maps any stable payoff $v \in \mathcal{S}$ to a payoff in $\mathcal{S}^{**}$. The process iteratively increases the payoffs of players on one side of the market to clear their legitimate objections through calibrated adjustments that do not create new legitimate objections on the other side. Once one side is free of objections, the process clears those of the other side, terminating at a stable payoff to which no player has a legitimate objection.

Our construction rests on expressing the no-blocking constraints among players in $N \setminus U$ as bounds on same-side payoff differences. For $i, i' \in I \setminus U$, the constraints along all sequences in $I \setminus U$ from i to i' reduce to a single bound $v_i - v_{i'} \leq C_{ii'}$, where the constant $C_{ii'}$ is the difference in coalitional worths from removing i rather than i'—alongside $\eta(i')$ and U—from the market. The bound binds exactly when i can be residually matched with $\eta(i')$. Its slack $C_{ii'} - (v_i - v_{i'})$ depends only on the payoffs of i and i', and is thus the increase in i's payoff at which that residual match becomes available, an *activation threshold*.

When a player $i \in I \setminus U$ has a legitimate objection to a payoff $v \in \mathcal{S}$, clearing it requires increasing her payoff while adjusting others' payoffs to preserve stability. For an admissible increment d in i's payoff, the smallest such adjustment increases the payoff of each player in $I \setminus U$ by the excess of d over her activation threshold (if positive), through a transfer from her efficient partner. The smallest increment that clears i's objection is determined by these thresholds and her OOP payoffs in the corresponding single-match games. The resulting mapping F^i is isotone, and $F^i(v)$ is the $\geq_I$-least stable payoff to which i does not object and that $\geq_I$-dominates v.

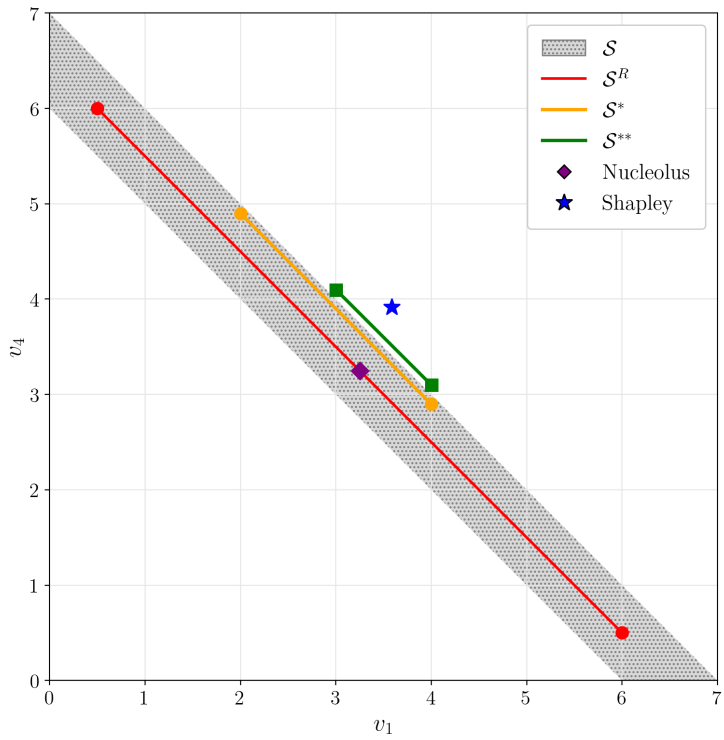

FIGURE 2. Projection of the four solution concepts, along with the nucleolus and the Shapley value, on the (v_1, v_4) plane ($\mathcal{S}^*$ and $\mathcal{S}^{**}$ segments are offset from the line $v_1 + v_4 = 7$ for visibility)

The reason these adjustments do not simply switch the problem from side I to side J is an *asymmetry of objections*. Any player j linked to i along a rematching chain can be left in a single-match game with i and the unmatched players. Since v is in the core and unmatched players receive zero payoffs, $v_i + v_j$ is at least the worth of the coalition $\{i, j\} \cup U$, which in turn is at least the sum of the OOP payoffs of i and j in that single-match game. Hence, when v_i falls short of i's OOP payoff, v_j exceeds j's OOP payoff by at least as much. Player j can therefore absorb the reduction needed to clear i's objection without acquiring a legitimate objection herself.

Repeatedly cycling the mappings F^i over all players $i \in I \setminus U$ increases the payoffs of players in I step by step, each time making the smallest adjustment consistent with stability that clears the targeted objection, as the remaining objections shift. This *tâtonnement process* converges in a finite number of steps, bounded uniformly over all stable payoffs, and yields an operator Φ^I that clears all objections on side I without creating new legitimate objections from players in J. As with F^i, the increase is minimal: $\Phi^I(v)$ is the lowest stable payoff to which no player in I objects and that $\geq_I$-dominates v. Defining Φ^J symmetrically, it follows that a single application of Φ^I followed by a single application of Φ^J takes any stable payoff into the refined solution. In particular, $\mathcal{S}^{**}$ is nonempty. Moreover, neither operator changes objection-free payoffs, so every element of $\mathcal{S}^{**}$ is a fixed point of $\Phi^J \circ \Phi^I$. The composition therefore projects $\mathcal{S}$ onto $\mathcal{S}^{**}$, and $\mathcal{S}^{**}$ equals its set of

fixed points. We show that $\mathcal{S}^{**}$ is also the set of fixed points of the single sweep $F^J \circ F^I$ (even though this composition does not necessarily map $\mathcal{S}$ into $\mathcal{S}^{**}$).

Our construction establishes that, like $\mathcal{S}$, the refinement $\mathcal{S}^{**}$ is a complete lattice, with meet and join characterized in terms of Φ^I and Φ^J, and I-optimal and J-optimal payoffs given by $\Phi^J(\bar{v}^I)$ and $\Phi^I(\bar{v}^J)$, respectively.[5] This lattice structure is intrinsic to $\mathcal{S}^{**}$ rather than inherited from $\mathcal{S}$: $(\mathcal{S}^{**}, \geq_I)$ need not be a sublattice of $(\mathcal{S}, \geq_I)$, and its meet and join are more complex than the componentwise ones of $(\mathcal{S}, \geq_I)$. Moreover, $\mathcal{S}^{**}$ need not be convex.

Following the Nash Program, we confirm that our cooperative refinement $\mathcal{S}^{**}$ emerges as the set of limit equilibrium payoffs of an intuitive non-cooperative game. Earlier non-cooperative treatments of the assignment game, by Kamecke (1989) and Pérez-Castrillo and Sotomayor (2002), devise demand games that implement the core or its side-optimal payoffs. The novelty of our non-cooperative approach is to transition to a bargaining endgame upon reaching a single-match subgame—a reasonable resolution once one side of the market shrinks to a single player. This sharpens the implementable set from the core to $\mathcal{S}^{**}$. A player who deviates to an aggressive demand risks being left unmatched or relegated to bilateral bargaining, which disciplines the implementable stable payoffs via the prospect of equal split or, more generally, divisions compatible with the OOP. The threat of relegation to bargaining is thus the non-cooperative counterpart of the *equity subject to no objection* principle that defines $\mathcal{S}^{**}$.

Our non-cooperative game consists of a *demand* phase followed (potentially) by a *bargaining* phase. Players submit demands in discrete rounds until they reach a single-match game or no more matches are possible. If a single-match game is reached, the remaining players *bargain* in bilateral meetings. The game involves two frictions: (1) a discount factor $\delta < 1$ applies between consecutive rounds, and (2) an additional discount factor $\beta < 1$ applies, once, upon the transition to the bargaining phase (reflecting the cost of decentralized matching). Bargaining proceeds according to one of the protocols discussed above.

We establish that the limit payoffs of subgame perfect equilibria (SPEs) satisfying two natural refinements belong to $\mathcal{S}^{**}$ as $(\delta, \beta) \to (1, 1)$. Conversely, for any $\beta < 1$, every payoff in $\mathcal{S}^{**}$ that is positive for all players matched under η is the payoff of an SPE satisfying the two refinements whenever δ is sufficiently close to 1.

The paper proceeds as follows. Section 2 reviews the assignment game and the core's lattice structure. Section 3 defines $\mathcal{S}^{**}$ and establishes preliminary results. Section 4 reformulates stability as payoff-difference bounds, constructs our operators and tâtonnement process, proves uniformly bounded termination, and derives the lattice and fixed-point characterizations of $\mathcal{S}^{**}$. Section 5 examines further structural properties of $\mathcal{S}^{**}$. In Section 6, we provide the non-cooperative implementation. Section 7 offers concluding remarks. Omitted proofs appear in the Appendix.

[5]Since Φ^I and Φ^J are isotone, Tarski's (1955) fixed-point theorem offers an alternative route to the lattice structure of $\mathcal{S}^{**}$. Our constructive approach delivers more: explicit formulae for the meet, join, lattice extremes, and minimal payoff increases needed to clear objections, as well as an algorithm that reaches $\mathcal{S}^{**}$ from any stable payoff in a uniformly bounded number of steps.

2. PRELIMINARIES

Let N denote the set of players, which is partitioned into two nonempty market sides, I and J. Suppose a *match* $(i,j) \in I \times J$ can generate a *value* $a_{ij} \geq 0$. It will be convenient to write $a_{ji} = a_{ij}$.

A *matching* is a function $\mu : N \to N \cup \{\emptyset\}$ such that for all $(i,j) \in I \times J$, we have $\mu(i) \in J \cup \{\emptyset\}$, $\mu(j) \in I \cup \{\emptyset\}$, and $\mu(i) = j$ if and only if $\mu(j) = i$. The condition $\mu(i) = j$ means that i is *matched with* j under μ, and thus i and j divide the value a_{ij}, while $\mu(k) = \emptyset$ means that player k is *not matched* under μ and receives zero payoff. A matching has an obvious interpretation as a set of disjoint matches $\{(i, \mu(i)) \mid i \in I, \mu(i) \neq \emptyset\}$. The *total value* generated by the matching μ is given by $\sum_{i \in I, \mu(i) \neq \emptyset} a_{i\mu(i)}$. A matching μ is *efficient* if it achieves the maximum total value.

We say that a payoff profile $(v_k)_{k \in N}$ is *stable* if the following conditions hold:

(i) $v_k \geq 0$ for all $k \in N$;
(ii) $v_i + v_j \geq a_{ij}$ for all $(i,j) \in I \times J$;
(iii) there exists a matching μ such that $\mu(k) = \emptyset \implies v_k = 0$, and $\mu(i) \neq \emptyset \implies v_i + v_{\mu(i)} = a_{i\mu(i)}$.

Let $\mathcal{S}$ denote the set of stable payoff profiles. Recall that $\mathcal{S}$ is the *core* of the *associated cooperative game* with transferable utility, in which the *worth* $W(N')$ of each coalition $N' \subseteq N$ is the maximum total value generated by a matching among its members.

We assume that match values are generic, in the sense that no two matchings, including the empty matching ($\mu(k) = \emptyset$ for every $k \in N$), generate the same total value.[6] Genericity then implies there is a unique *efficient matching* η. It also yields $a_{ij} > 0$ for all $(i,j) \in I \times J$. The empty matching has value 0, while the matching consisting of the single match (i,j) creates value a_{ij}; thus, $a_{ij} \neq 0$.

Let U denote the set of players *not matched* under η. It follows that either $U \subset I$ or $U \subset J$. For $K \in \{I, J\}$, let $K^* = K \setminus U$ denote the set of players in K matched under η. Let $N^* = I^* \cup J^*$ be the set of all matched players, and let $m := |I^*| = |J^*|$ denote the number of pairs matched under η. Under our genericity assumption, stability admits the following straightforward characterization in terms of the efficient matching η.

Lemma 1. *A payoff profile $(v_k)_{k \in N}$ is stable if and only if the following two conditions hold jointly:*

(i) $v_k \geq 0$ for all $k \in N$, with equality if $k \in U$;
(ii) $v_i + v_j \geq a_{ij}$ for all $(i,j) \in I \times J$, with equality if $j = \eta(i)$.

Stable payoff profiles can be compared from the perspective of each side of the market, and the resulting partial orders endow $\mathcal{S}$ with its familiar lattice structure. By Lemma 1, every $v \in \mathcal{S}$ satisfies the feasibility conditions $v_i + v_{\eta(i)} = a_{i\eta(i)}$ for $i \in I^*$ and $v_u = 0$ for $u \in U$. We say that a payoff profile $v \in \mathbb{R}^N$ is *feasible* if it satisfies these two conditions, and denote by $\mathcal{F}$ the set of feasible payoff profiles; thus $\mathcal{S} \subset \mathcal{F}$. Fix a side $K \in \{I, J\}$, and let L denote the opposite side.

[6] Note that this property is inherited by "subgames" ensuing after some pairs of players are matched.

For $v, w \in \mathcal{F}$, write $v \geq_K w$ if $v_k \geq w_k$ for all $k \in K$, and $v >_K w$ if in addition $v \neq w$. Two feasible payoff profiles that agree for players in K must agree for all players, so $\geq_K$ is antisymmetric. Since $\geq_K$ is also reflexive and transitive, it is a partial order on $\mathcal{F}$. Feasibility makes the two orders dual: $v \geq_I w \iff w \geq_J v$ (and likewise for the strict relations), so a result for one side carries over to the other.

The partially ordered set $(\mathcal{F}, \geq_K)$ is a lattice, with join $\bigvee^K$ and meet $\bigwedge^K$ determined componentwise: for any nonempty bounded subset $X \subseteq \mathcal{F}$,

$$(\textstyle\bigvee^K X)_k = \sup_{x \in X} x_k \qquad \text{and} \qquad (\textstyle\bigwedge^K X)_k = \inf_{x \in X} x_k, \qquad \forall k \in K,$$

with the L components pinned down by feasibility: $(\bigvee^K X)_l = \inf_{x \in X} x_l$ and $(\bigwedge^K X)_l = \sup_{x \in X} x_l$ for all $l \in L$, so that $\bigvee^K X, \bigwedge^K X \in \mathcal{F}$. These componentwise operators make no reference to stability and play a special role in the analysis. By construction, $\bigvee^I X = \bigwedge^J X$ and $\bigwedge^I X = \bigvee^J X$.

As discussed in Shapley and Shubik (1971) and Roth and Sotomayor (1990), for every nonempty $X \subseteq \mathcal{S}$, the profiles $\bigvee^K X$ and $\bigwedge^K X$ are stable. Consequently, $(\mathcal{S}, \geq_K)$ is a sublattice of $(\mathcal{F}, \geq_K)$ and forms a complete lattice.[7] The join $\bigvee^K_{\mathcal{S}} X$ and the meet $\bigwedge^K_{\mathcal{S}} X$ of X in $(\mathcal{S}, \geq_K)$ are given by the componentwise operators: $\bigvee^K_{\mathcal{S}} X = \bigvee^K X$ and $\bigwedge^K_{\mathcal{S}} X = \bigwedge^K X$.

The fact that the join and meet under $(\mathcal{S}, \geq_K)$ are determined componentwise by $\bigvee^K$ and $\bigwedge^K$ is a *special property* of $\mathcal{S}$ and does not, in general, extend to refinements of $\mathcal{S}$. We show that our main refinement $\mathcal{S}^{**}$ is also a complete lattice but need not be a *sublattice* of $\mathcal{S}$.

As $\mathcal{S}$ is a complete lattice under $\geq_K$, it has two extremes, $\bar{v}^K := \bigvee^K_{\mathcal{S}} \mathcal{S} = \bigvee^K \mathcal{S}$ and $\underline{v}^K := \bigwedge^K_{\mathcal{S}} \mathcal{S} = \bigwedge^K \mathcal{S}$. By duality, $\bar{v}^K = \underline{v}^L$. For our purposes, we will say that a function $f : \mathcal{S} \to \mathcal{F}$ is *isotone* if $v \geq_K w$ implies $f(v) \geq_K f(w)$; by the duality of the orders $\geq_I$ and $\geq_J$, this property is independent of K.

3. The Main Refinement: Reassignments and Legitimate Objections

The definition of $\mathcal{S}^*$ in the introduction is based on the idea that if $v_k > a_{k\eta(k)}/2$ at a stable payoff v, then player k's larger share must be justified via a "binding" outside option. But it does not explain why $\eta(k)$ might not effectively *object*. If $\eta(k)$ were to make herself unavailable to match with k, the resulting "rematching" triggered by k exercising her outside option might well be favorable for player $\eta(k)$, as illustrated by the example from Figure 1 discussed in the introduction.

A more complete cooperative theory requires understanding how objections might restructure matches, and how that would ultimately affect the objecting player's outcome. An objection should not stand if the objecting player is not better off when we consider potential rematching. In the example from Figure 1, we argued that the stable payoff given by $w = (3.5, 6.5, 4.5, 3.5) \in \mathcal{S}^*$

[7]Recall that for a lattice $(P, \succeq)$ with join $\bigvee_P$ and meet $\bigwedge_P$, a subset P' of P is a *sublattice* if it is closed under the join and meet of $(P, \succeq)$: $\bigvee_P\{a, b\},\ \bigwedge_P\{a, b\} \in P'$ for every $a, b \in P'$. The lattice $(P, \succeq)$ is *complete* if every subset of P has a least upper bound (join) and a greatest lower bound (meet) in P.

is not vulnerable to objections from player 1, who obtains less than an equal share in her efficient match. We seek to identify those stable payoffs that are not vulnerable to hypothetical objections.

Our approach to countering objections is conservative in the sense that we consider rematching that substantially preserves the original payoff proposal v. Suppose player $i \in I$ objects to v. If $i \in U$, then no rematching is necessary and the objection is futile. Suppose now that $i \in I^*$. Then, one possible response to i's objection is to implement all matches other than $(i, \eta(i))$ according to the payoff divisions under v, and leave i in the subgame with $\eta(i)$ and the players in U. More generally, it might be feasible to maintain the payoffs of all $k \in N^*$ with the exception of the objecting player i and at most one player $j \in J^*$. Indeed, if the players of N^* who maintain their payoffs are matched among themselves, then equal numbers of players from the two sides must become unmatched: unmatching the objector $i \in I^*$ requires that exactly one player $j \in J^*$ become unmatched as well. We conservatively require that "rematching" in response to i's objection modifies (v, η) and maintains payoffs for all players except i and one $j \in J^*$. This leaves players i and j, together with the unmatched players U, in a single-match assignment game $\Gamma_{ij} := \{i, j\} \cup U$. Given our arguments for selecting the OOP prediction in these single-match games, we assume that player i's objection will result in the OOP outcome in Γ_{ij}. We write $\omega_k(\Gamma_{ij})$ for player k's payoff in Γ_{ij} under the OOP (extrapolated from the formulae (1) for the representative single-match game with $|I| = 1$ discussed in the introduction). We call a rematching of this kind, together with its associated payoffs, an ij-*reassignment* under v.

Definition 1 (Reassignments)**.** Suppose $v \in \mathcal{S}$. For $i \in I^*, j \in J^*$, we say that (v', η') is an ij-*reassignment* under v if

- η' is a matching such that

$$\forall k \in N^* \setminus \{i, j\} : \eta'(k) \in N^* \setminus \{i, j\}, v'_k = v_k \text{ and } v_k + v_{\eta'(k)} = a_{k\eta'(k)};$$

- for players in Γ_{ij}, v' and η' conform to the OOP in the single-match game Γ_{ij}.

Given the symmetry between i and j in the two conditions above, we also refer to (v', η') as a ji-*reassignment* when the conditions are satisfied.

In words, an ij-reassignment under v is a matching η' and a payoff vector v' feasible under η' such that all players matched under η with the exception of i and j are matched with one another and receive their payoffs under v. Furthermore, players in the "residual" single-match "subgame" Γ_{ij} have v' payoffs and η' matches that conform to the OOP applied to Γ_{ij}. Note that an $i\eta(i)$-reassignment (v', η') always exists, and in this case the definition implies that $\eta'(i) = \eta(i)$ (since η is the unique efficient matching in the assignment game, $i\eta(i)$ is the unique efficient match in the single-match game $\Gamma_{i\eta(i)}$).

We say that an ij-reassignment *involves* players i and j. The next definition posits that a player $k \in N^*$ cannot effectively object to a proposed stable payoff v if there is a reassignment involving her under v in which she obtains a payoff not exceeding v_k.

Definition 2 (Legitimate Objections)**.** Player $k \in N^*$ has a *legitimate objection* to $v \in \mathcal{S}$ if $v_k < v'_k$ for every reassignment (v', η') under v involving k. Let $\mathcal{I}(v)$ and $\mathcal{J}(v)$ denote the set of players in I^* and J^*, respectively, who have a legitimate objection to v.

The quantifier in the definition is never vacuous: the $k\eta(k)$-reassignment in which every player in $N^* \setminus \{k, \eta(k)\}$ retains her match under η always exists.

We can now introduce our main cooperative solution concept.

Definition 3 (Main Refinement)**.** $\mathcal{S}^{**}$ is the set of stable payoffs v with the property that no player $k \in N^*$ has a legitimate objection to v. That is, $\mathcal{S}^{**} = \{v \in \mathcal{S} \mid \mathcal{I}(v) = \mathcal{J}(v) = \emptyset\}$.

It is not difficult to see that $\mathcal{S}^{**}$ is a refinement of $\mathcal{S}^*$: justifying a less-than-equal share to a player $k \in N^*$ via a reassignment involving k automatically leads to $\eta(k)$'s more-than-equal share being equal to a binding outside option (as required by the definition of $\mathcal{S}^*$).

Lemma 2 ($\mathcal{S}^{**}$ Is a Refinement of $\mathcal{S}^*$)**.** $\mathcal{S}^{**} \subseteq \mathcal{S}^*$.

The proof of Lemma 2 and other proofs omitted in the main body of the paper appear in the Appendix.

Figure 2 in the introduction illustrates the four solution concepts discussed here—the set of stable payoffs, Rochford's solution, and our two refinements of stability—in the assignment game from Figure 1. Since $v_2 = 10 - v_4$ and $v_3 = 8 - v_1$ for every $v \in \mathcal{S}$, we identify each stable payoff with its projection (v_1, v_4) on the plane depicted in Figure 2. In these coordinates,

$$\begin{aligned}
\mathcal{S} &= \{(v_1, v_4) \mid v_1, v_4 \geq 0,\ v_1 + v_4 \in [6, 7]\} \\
\mathcal{S}^R &= \{(v_1, v_4) \mid v_1 + v_4 = 6.5,\ v_1 \in [0.5, 6]\} \\
\mathcal{S}^* &= \{(v_1, v_4) \mid v_1 + v_4 = 7,\ v_1 \in [2, 4]\} \\
\mathcal{S}^{**} &= \{(v_1, v_4) \mid v_1 + v_4 = 7,\ v_1 \in [3, 4]\}.
\end{aligned}$$

The nucleolus and the Shapley value shown in Figure 2 are given by $(v_1, v_4) = (3.25, 3.25)$ and $(v_1, v_4) = (43/12, 47/12)$, respectively. The nucleolus belongs to the kernel $\mathcal{S}^R$, whereas the Shapley value falls outside the core $\mathcal{S}$.[8]

We next develop the tools for identifying reassignments under a given stable payoff. If an ij-reassignment under v exists then we refer to ij as a *residual pair* under v. It is the pair whose payoffs are determined in the "residual" single-match game Γ_{ij}.

Definition 4 (Residual Pairs)**.** For $v \in \mathcal{S}$, $\mathcal{R}(v) \subseteq \{\{i, j\} \mid i \in I^*,\ j \in J^*\}$ denotes the set of pairs for which there exists an ij-reassignment under v. We write $ij \in \mathcal{R}(v)$ and $ji \in \mathcal{R}(v)$ for $\{i, j\} \in \mathcal{R}(v)$.

[8]In assignment games with $U \neq \emptyset$, the Shapley value also assigns positive payoffs to unmatched players, which is hard to justify in the assignment game, where value is created and divided within matches.

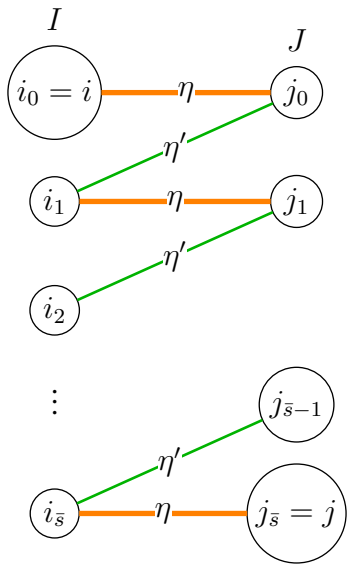

FIGURE 3. Rematching chain induced by a residual pair $ij \in \mathcal{R}(v)$.

Recall that for every $v \in \mathcal{S}$ and $i \in I^*$, there exists an $i\eta(i)$-reassignment under v, so $i\eta(i) \in \mathcal{R}(v)$. In terms of residual pairs, the sets $\mathcal{I}(v)$ and $\mathcal{J}(v)$ underlying $\mathcal{S}^{**}$ can be succinctly summarized as follows:

$$\begin{aligned} \mathcal{I}(v) &= \{i \in I^* \mid \forall j \in J^*,\ ij \in \mathcal{R}(v) \implies v_i < \omega_i(\Gamma_{ij})\} \\ \mathcal{J}(v) &= \{j \in J^* \mid \forall i \in I^*,\ ij \in \mathcal{R}(v) \implies v_j < \omega_j(\Gamma_{ij})\}. \end{aligned}$$

The residual pairs in $\mathcal{R}(v)$ admit a simple characterization in terms of the rematching chains defined next.

Definition 5 (Rematching Chains)**.** A *rematching chain* under v is a sequence of distinct players $(k_s)_{s=0}^{\bar{s}}$ in K^* (where $K^* \in \{I^*, J^*\}$) with $\bar{s} \geq 0$ such that[9]

$$v_{k_{s+1}} + v_{\eta(k_s)} = a_{k_{s+1}\eta(k_s)} \text{ for } s = 0, 1, \ldots, \bar{s} - 1. \tag{2}$$

Figure 3 illustrates a subset of players involved in a rematching chain $i_0, \ldots, i_{\bar{s}}$ corresponding to an ij-reassignment (v', η'). There is a tight connection between elements of $\mathcal{R}(v)$ and rematching chains under v. We show that for $k, k' \in K^*$ with $k\eta(k') \in \mathcal{R}(v)$, every $k\eta(k')$-reassignment (v', η') under v induces a rematching chain $k_0 = k, \ldots, k_{\bar{s}} = k'$ under v, where $k_{s+1} = \eta'(\eta(k_s))$ for $s = 0, 1, \ldots, \bar{s}-1$; conversely, the endpoints k and k' of any rematching chain under v generate a pair $k\eta(k')$ in $\mathcal{R}(v)$.[10] Lemma 3 states this result for chains in I. *For brevity, we state many of our results for one side of the market; symmetric statements hold for the other side. We will be explicit when we invoke the symmetric version.*

[9]For $\bar{s} = 0$, condition (2) is vacuous.

[10]For $\bar{s} = 0$, we have $k = k'$, and as already argued, $k\eta(k) \in \mathcal{R}(v)$.

Lemma 3 (Residual Pairs and Rematching Chains)**.** *For $i \in I^*, j \in J^*$, the pair ij belongs to $\mathcal{R}(v)$ if and only if there exists a rematching chain $(i_s)_{s=0}^{\bar{s}}$ under v with $i_0 = i$ and $i_{\bar{s}} = \eta(j)$. Moreover, if $ij \in \mathcal{R}(v)$ and $v_{i'} + v_j = a_{i'j}$ for $(i', j) \in I^* \times J^*$, then $i\eta(i') \in \mathcal{R}(v)$.*

Legitimate objections have an asymmetry that underlies the construction of the next section. Consider a payoff $v \in \mathcal{S}$, players $i \in I^*$ and $j \in J^*$, and the single-match game $\Gamma_{ij} = \{i, j\} \cup U$. The coalition Γ_{ij} does not block v because v is in the core. Since $v_u = 0$ for all $u \in U$ and the OOP payoffs are feasible in Γ_{ij}, it follows that

$$v_i + v_j \geq W(\Gamma_{ij}) \geq \omega_i(\Gamma_{ij}) + \omega_j(\Gamma_{ij}).$$

Thus, if v_i falls short of i's OOP payoff in Γ_{ij}, then v_j exceeds j's OOP payoff by at least the same amount.

Lemma 4 (Asymmetry of Objections)**.** *Suppose that $v \in \mathcal{S}$, $i \in I^*$, $j \in J^*$ and $\omega_i(\Gamma_{ij}) > v_i$. Then, the following conditions hold:*

(i) $v_j - \omega_j(\Gamma_{ij}) \geq \omega_i(\Gamma_{ij}) - v_i > 0$*;*
(ii) $\omega_j(\Gamma_{ij}) \geq a_{uj}$ *for all* $u \in U \cap I$*; in particular,* $v_j - a_{uj} \geq \omega_i(\Gamma_{ij}) - v_i$*.*

By part (i), if $i \in \mathcal{I}(v)$ and $ij \in \mathcal{R}(v)$, then $v_j > \omega_j(\Gamma_{ij})$, so $j \notin \mathcal{J}(v)$. Thus, a player who can be residually matched with an objector never has a legitimate objection.

In the next section, we present a tâtonnement process that transforms any stable payoff into a payoff in $\mathcal{S}^{**}$; in particular, $\mathcal{S}^{**}$ is nonempty. Sets of stable payoffs free of objections from a particular player i, or more broadly from all players in I, play an important role in the analysis. Let

$$\begin{aligned} \mathcal{S}^{**i} &= \{w \in \mathcal{S} \mid i \notin \mathcal{I}(w)\} \\ \mathcal{S}^{**I} &= \{w \in \mathcal{S} \mid \mathcal{I}(w) = \emptyset\} \end{aligned}$$

denote the sets of stable payoffs to which player $i \in I^*$ does not have an objection and to which no player in I has an objection, respectively (with $\mathcal{S}^{**j}$ and $\mathcal{S}^{**J}$ defined symmetrically).

4. Clearing Objections and the Lattice Structure of $\mathcal{S}^{**}$: A Constructive Approach

Starting from a stable payoff $v \in \mathcal{S}$, we modify v incrementally to eliminate legitimate objections, first on one side of the market, say I, and then on the other. Our procedure clears the objections of players in I by reducing, in a calibrated way, the payoffs of players in J. Maintaining stability generally requires increasing the payoffs of other players in I linked to an objector i along a rematching chain, at the expense of their efficient partners in J. Increasing i's payoff also makes stability constraints bind and thereby creates new rematching chains starting at i. Player i's objection is cleared once her payoff is at least her OOP payoff in the single-match game with some residual partner, possibly one made available by a new chain. The payoffs of players in I increase

monotonically until all their objections are cleared, and the set of players in J with legitimate objections weakly declines at every step. The increases are minimal, in that the procedure terminates at the lowest stable payoff $\geq_I$-dominating v to which no player in I objects. We then clear the objections of players in J similarly, and the asymmetry of objections ensures that no player in I has a legitimate objection at any step.

The construction proceeds in several steps. Section 4.1 rewrites the no-blocking constraints among players in N^* as bounds on same-side payoff differences. Section 4.2 uses these bounds to construct an operator F^i that clears the objection of player $i \in I^*$ with the minimal increase in side I payoffs consistent with stability. Section 4.3 shows that iterating F^i over $i \in I^*$ from any $v \in \mathcal{S}$ terminates in finitely many steps at a payoff $\Phi^I(v) \in \mathcal{S}^{**I}$, and establishes existence via $\Phi^J(\Phi^I(v)) \in \mathcal{S}^{**}$. Section 4.4 derives the lattice structure of $\mathcal{S}^{**}$, and Section 4.5 provides fixed-point characterizations for the solution.

4.1. **Stability as Bounds on Same-Side Payoff Differences.** For $w \in \mathcal{F}$ and a pair $(i,j) \in I \times J$, let

$$\sigma_{ij}(w) = w_i + w_j - a_{ij}$$

denote the stability *slack* of the pair. The payoff profile w is stable when no slack is negative and the remaining conditions of Lemma 1 hold. Fix $i, i' \in I^*$. For any sequence of distinct players $i = i_0, \ldots, i_{\bar{s}} = i'$ in I^*, taking into account that $w_{i_s} + w_{\eta(i_s)} = a_{i_s\eta(i_s)}$, the total slack telescopes:

$$\sum_{s=0}^{\bar{s}-1} \sigma_{i_{s+1}\eta(i_s)}(w) = w_{i'} - w_i + \sum_{s=0}^{\bar{s}-1} \left[a_{i_s\eta(i_s)} - a_{i_{s+1}\eta(i_s)}\right]. \tag{3}$$

The payoffs of the intermediate players drop out, and what remains is the difference in the payoffs of i and i' plus a term that depends on match values alone. Minimizing that term over sequences yields a single constant for each pair $(i,i') \in I^* \times I^*$.

The minimum admits a formula in terms of coalitional worths W in the associated cooperative game. The pairs $(i_{s+1}, \eta(i_s))$ along a sequence from i to i', together with the η matches of the players in I^* outside the sequence, form a matching of $N^* \setminus \{i, \eta(i')\}$, and the match-value term in (3) equals $W(N^*) - a_{i'\eta(i')}$ minus the total value of this matching. Minimizing the term over sequences thus amounts to maximizing the value of the induced matching, which is bounded above by $W(N^* \setminus \{i, \eta(i')\})$. We show that some sequence induces a matching that attains this bound. The minimum of the match-value term is therefore

$$C_{ii'} := W(N^*) - W(N^* \setminus \{i, \eta(i')\}) - a_{i'\eta(i')}$$

for $i, i' \in I^*$, and the least total slack in (3) under $w \in \mathcal{F}$ is this constant plus the payoff difference,

$$\tau_{ii'}(w) := w_{i'} - w_i + C_{ii'}.$$

The next result confirms this value of the least total slack, characterizes residual pairs in terms of τ, and shows that a residual pair ij persists when the payoffs of players in I other than i decrease while i's payoff remains unchanged.

Lemma 5 (Stability Slack and Residual Pairs)**.** *For every $w \in \mathcal{F}$ and all $i, i' \in I^*$:*

(i) $\tau_{ii'}(w) = \min\Big\{ \sum_{s=0}^{\bar{s}-1} \sigma_{i_{s+1}\eta(i_s)}(w) \;\Big|\; i = i_0, \ldots, i_{\bar{s}} = i' \text{ distinct players in } I^* \Big\}$;

(ii) if $w \in \mathcal{S}$, then $i\eta(i') \in \mathcal{R}(w)$ if and only if $\tau_{ii'}(w) = 0$;

(iii) if $w, w' \in \mathcal{S}$, $w \geq_I w'$, $w_i = w'_i$, and $ij \in \mathcal{R}(w)$ for some $j \in J^*$, then $ij \in \mathcal{R}(w')$ and $w'_j = w_j$.

If $w \in \mathcal{S}$, then $\tau_{ii'}(w) \geq 0$ by part (i). With the payoff of i' fixed, $\tau_{ii'}$ decreases one for one with i's payoff. Thus, provided stability is maintained, part (ii) implies that $\tau_{ii'}(w)$ is the increase in i's payoff at which $i\eta(i')$ becomes residual. We call $\tau_{ii'}(w)$ the *activation threshold* of the pair $i\eta(i')$ at w.

No pair in $I^* \times J^*$ blocks $w \in \mathcal{F}$ if and only if $\sigma_{i'\eta(i)}(w) \geq 0$ for all $i \neq i' \in I^*$. By Lemma 5.(i), the latter condition is equivalent to $\tau_{ii'}(w) \geq 0$, that is, $w_i - w_{i'} \leq C_{ii'}$, for all $i, i' \in I^*$.

Lemma 6 (No-Blocking Constraints among Players in N^*)**.** *A payoff profile $w \in \mathcal{F}$ is blocked by no pair in $I^* \times J^*$ if and only if $w_i - w_{i'} \leq C_{ii'}$ for all $i, i' \in I^*$.*

The following lemma collects the properties of the constants C used in the sequel.

Lemma 7 (Properties of the Constants C)**.** *For all $i, i', i'' \in I^*$:*

(i) $C_{ii} = 0$ *and* $C_{ii'} = W(N^* \setminus \{i', \eta(i')\}) - W(N^* \setminus \{i, \eta(i')\})$;

(ii) $C_{ii''} \leq C_{ii'} + C_{i'i''}$.

By part (i), $C_{ii'}$ is the difference in coalitional worths from removing i rather than i' from the market, in each case alongside $\{\eta(i')\} \cup U$. Part (ii) reflects the fact that concatenating a sequence from i to i' with one from i' to i'' yields, after deleting repetitions, a sequence from i to i''.

4.2. **Clearing Player i's Objection.** Fix $v \in \mathcal{S}$ and $i \in I^*$, and consider increasing v_i gradually while preserving stability without reducing the payoff of any player in I. Increasing v_i reduces the payoff of $\eta(i)$ by the same amount, which tightens the stability constraints between $\eta(i)$ and other players in I^*. Once such a constraint binds, say for the pair $(i', \eta(i))$, maintaining stability requires increasing the payoff of i' at the expense of $\eta(i')$. The adjustment thus propagates along sequences of players in I^* starting at i. By (3), as long as the payoff of i' is unchanged, the total slack along every sequence from i to i' decreases one for one with the increase in v_i, regardless of how the payoffs of intermediate players change. By Lemma 5.(ii), the pair $i\eta(i')$ becomes residual when the increase reaches the least total slack $\tau_{ii'}(v)$ (provided the adjustment preserves stability). Since $\tau_{ii'}(v)$ is the length of a shortest path from i to i' when edge weights are given by stability slacks, residual partners become available to i in increasing order of their activation thresholds.

For a given *increment* $d \geq 0$ in i's payoff, we seek payoff profiles in the set

$$\mathcal{D}^{i,d}(v) := \{w \in \mathcal{S} \mid w \geq_I v,\ w_i \geq v_i + d\}.$$

For every $w \in \mathcal{D}^{i,d}(v)$ and $i' \in I^*$, Lemma 6 yields $w_{i'} \geq w_i - C_{ii'} \geq v_i + d - C_{ii'}$. Together with $w_{i'} \geq v_{i'}$, this leads to

$$w_{i'} \geq \max\{v_{i'},\, v_i + d - C_{ii'}\} = v_{i'} + \max\{d - \tau_{ii'}(v), 0\} \qquad \text{for every } i' \in I^*.$$

The next definition sets the payoff of every player in I^* equal to its lower bound.

Definition 6 (The $F^{i,d}$ Operators)**.** For $v \in \mathcal{S}$, $i \in I^*$ and $d \geq 0$, define $F^{i,d}(v) \in \mathcal{F}$ by

$$F^{i,d}_{i'}(v) = \max\{v_{i'},\, v_i + d - C_{ii'}\} = v_{i'} + \max\{d - \tau_{ii'}(v), 0\} \qquad \text{for } i' \in I^*,$$

with the remaining payoffs determined by feasibility ($F^{i,d}_{\eta(i')}(v) = a_{i'\eta(i')} - F^{i,d}_{i'}(v)$ for $i' \in I^*$, and $F^{i,d}_u(v) = 0$ for $u \in U$).

Since $\tau_{ii}(v) = 0$ by Lemma 7.(i), $F^{i,d}(v)$ increases i's payoff by d. In general, $F^{i,d}(v)$ increases the payoff of each player $i' \in I^*$ by the excess of d over her activation threshold $\tau_{ii'}(v)$ (if positive) via a transfer from her efficient partner. The following lemma provides a simple test for the stability of $F^{i,d}(v)$, establishes its $\geq_I$-minimality, and characterizes activation thresholds via stable payoff adjustments.

Lemma 8 (Stability and Minimality of $F^{i,d}$)**.** *Fix $v \in \mathcal{S}$, $i \in I^*$, and $d \geq 0$.*

(i) $F^{i,d}(v) \in \mathcal{S}$ if and only if

$$F^{i,d}_j(v) \geq \max\Big\{0,\, \max_{u \in U \cap I} a_{uj}\Big\} \qquad \textit{for every } j \in J^*.$$

(ii) $\mathcal{D}^{i,d}(v) \neq \emptyset$ if and only if $F^{i,d}(v) \in \mathcal{S}$, in which case

$$F^{i,d}(v) = \textstyle\bigwedge^I \mathcal{D}^{i,d}(v).$$

(iii) For every $i' \in I^*$ such that $\mathcal{D}^{i,\tau_{ii'}(v)}(v) \neq \emptyset$,

$$\tau_{ii'}(v) = \max\big\{w_i - v_i \;\big|\; w \in \mathcal{S},\; w \geq_I v,\; w_{i'} = v_{i'}\big\}.$$

By Lemma 5.(i), the activation threshold $\tau_{ii'}(v)$ is the minimum total stability slack over sequences from i to i'. Part (iii) provides a *complementary optimization characterization*: whenever $\mathcal{D}^{i,\tau_{ii'}(v)}(v) \neq \emptyset$, this threshold is the largest increase in i's payoff achievable while preserving stability, weakly increasing all side I payoffs, and keeping i''s payoff unchanged.

We now determine the smallest d that *clears i's objection*, and show that the resulting adjustment creates no new objection on the J side. Suppose that $F^{i,d}(v) \in \mathcal{S}$. Then, player i does not have a legitimate objection to $F^{i,d}(v)$ if and only if there exists $i' \in I^*$ such that $i\eta(i') \in \mathcal{R}(F^{i,d}(v))$ and

$$F^{i,d}_i(v) = v_i + d \geq \theta_{ii'} := \omega_i(\Gamma_{i\eta(i')}).$$

By Definition 6, $\tau_{ii'}(F^{i,d}(v)) = \max\{\tau_{ii'}(v) - d, 0\}$, and Lemma 5.(ii) yields

$$i\eta(i') \in \mathcal{R}(F^{i,d}(v)) \iff d \geq \tau_{ii'}(v). \tag{4}$$

Combined with the condition $v_i + d \geq \theta_{ii'}$ above, player i' therefore provides i with a residual partner $\eta(i')$ that clears her objection at $F^{i,d}(v)$ if and only if

$$d \geq \max\{\tau_{ii'}(v),\, \theta_{ii'} - v_i\}.$$

The minimal increment d that clears i's objection is thus

$$\Delta^i(v) := \min_{i' \in I^*} \max\{\tau_{ii'}(v),\, \theta_{ii'} - v_i\}.$$

Definition 7 (The F^i Operator)**.** For $v \in \mathcal{S}$ and $i \in I^*$, define

$$F^i(v) := F^{i,\Delta^i(v)}(v).$$

Substituting the expression of $\tau_{ii'}(v)$ and using Definition 6, we obtain

$$F_i^i(v) = v_i + \Delta^i(v) = \min_{i' \in I^*} \max\{v_{i'} + C_{ii'},\, \theta_{ii'}\} \quad \text{and} \quad F_{i'}^i(v) = \max\{v_{i'},\, F_i^i(v) - C_{ii'}\}, \tag{5}$$

with the payoffs of the remaining players pinned down by feasibility.

The next result confirms that $F^i(v)$ is stable and establishes other essential properties of F^i on which the rest of the construction builds. Part (iii) shows that $F^i(v)$ is the lowest stable payoff that $\geq_I$-dominates v and leaves i without a legitimate objection. Part (iv) states that F^i creates no new objectors in J. This fact is crucial for composing operators associated with opposite sides of the market: it allows us to clear objections on one side without eliciting new objections from the other.

Theorem 1 (Properties of F^i)**.** *For every $i \in I^*$, the function F^i is isotone, and for every $v \in \mathcal{S}$:*

(i) $F^i(v) \in \mathcal{S}$*;*
(ii) $F^i(v) \geq_I v$*, with equality if and only if* $i \notin \mathcal{I}(v)$*;*
(iii) $F^i(v) \in \mathcal{S}^{**i}$ *and* $F^i(v) = \bigwedge^I \{w \in \mathcal{S}^{**i} \mid w \geq_I v\}$*;*
(iv) $\mathcal{J}(F^i(v)) \subseteq \mathcal{J}(v)$*.*

To develop intuition for part (iv), consider a player $j \in J$ whose payoff strictly decreases under $F^i(v)$, and let $i' := \eta(j) \in I^*$. Then,

$$0 \leq \tau_{ii'}(v) < \Delta^i(v) \leq \theta_{ii'} - v_i.$$

Hence, the activation threshold of i' at v is below $\Delta^i(v)$, and (4) implies that $ij \in \mathcal{R}(F^i(v))$. Since $v_i < \theta_{ii'}$, i's payoff falls short of her OOP payoff in the single-match game Γ_{ij} associated with the residual pair ij at $F^i(v)$. Lemma 4.(i) implies that $v_j - \omega_j(\Gamma_{ij}) \geq \theta_{ii'} - v_i$. Since $\theta_{ii'} - v_i \geq \Delta^i(v)$ and $F^i(v)$ reduces j's payoff by $\Delta^i(v) - \tau_{ii'}(v)$, we have that $F_j^i(v) \geq \omega_j(\Gamma_{ij})$, so j does not have a legitimate objection at $F^i(v)$. For any player in J whose payoff remains unchanged under $F^i(v)$, the symmetric version of Lemma 5.(iii) ensures that any residual pair that prevented her from objecting at v remains residual at $F^i(v)$, so she cannot acquire a new objection.

A first consequence of parts (i) and (ii) of the theorem is that no player in I objects to the I-optimal stable payoff. Indeed, $i \in \mathcal{I}(\bar{v}^I)$ would imply that $F^i(\bar{v}^I) \in \mathcal{S}$ and $F^i(\bar{v}^I) >_I \bar{v}^I$, contradicting $\bar{v}^I = \bigvee^I \mathcal{S}$.

Corollary 1. $\bar{v}^I \in \mathcal{S}^{**I}$.

4.3. **Clearing the Objections of Side I.** We compose the player-targeted operators into a single sweep F^I through I^*.

Definition 8 (The F^I Operator)**.** Fix an arbitrary ordering $(i_1, \ldots, i_m)$ of I^*, and define

$$F^I := F^{i_m} \circ \ldots \circ F^{i_1}.$$

A single application of F^I cycles over the players in I^* in the fixed order $(i_1, \ldots, i_m)$, applying F^{i_q} at each stage. Since clearing player i_q's objection may create objections for players processed earlier, a single application of F^I need not generate a payoff in $\mathcal{S}^{**I}$. We therefore iterate F^I. Repeated application of parts (i) and (ii) of Theorem 1 implies that the sequence $\left((F^I)^{(z)}(v)\right)_{z\geq 0}$ lies in $\mathcal{S}$ and is weakly increasing with respect to $\geq_I$. Since the sequence is bounded above by $\bar{v}^I$, it converges. We establish a stronger result: the sequence becomes constant after a number of iterations bounded uniformly over $v \in \mathcal{S}$.

Theorem 2 (Uniform Finite Termination)**.** *There is an integer Z, depending only on the assignment game, such that*

$$(F^I)^{(z)}(v) = (F^I)^{(Z)}(v) \quad \textit{for every } v \in \mathcal{S} \textit{ and every } z \geq Z.$$

The proof exploits the closed form (5): a strict update of player i's payoff at the current iterate w moves it either to an objection threshold or to an activation level $w_i + \tau_{ii'}(w) = w_{i'} + C_{ii'}$. Tracing strict updates back to earlier updates, each repetition of a player entails a strictly positive payoff increase drawn from a finite set determined by match values via the constants $C_{ii'}$. Since the total payoff of players in I is bounded, the number of such repetitions, and hence the length of each backward sequence, is uniformly bounded.

The tâtonnement process iterating F^I from any $v \in \mathcal{S}$ is thus an exact algorithm that reaches a fixed point of F^I in at most Z sweeps, stopping as soon as a sweep leaves the payoff profile unchanged. Although the bound Z developed in the proof of Theorem 2 is coarse, it allows us to define the outcome of the process as a fixed finite iterate, yielding our main operator Φ^I.

Definition 9 (The Φ^I Operator)**.** For every $v \in \mathcal{S}$, define

$$\Phi^I(v) := (F^I)^{(Z)}(v).$$

The operator Φ^J is defined symmetrically, repeatedly cycling through the player-targeted operators F^j for $j \in J^*$.

Because the iteration defining Φ^I terminates after finitely many sweeps, the properties of the mappings F^i carry over to Φ^I.

Corollary 2 (Properties of Φ^I)**.** *The function Φ^I is isotone, and for every $v \in \mathcal{S}$:*

(i) $\Phi^I(v) \in \mathcal{S}^{**I}$*;*

(ii) $\Phi^I(v) \geq_I v$*, with equality if and only if* $v \in \mathcal{S}^{**I}$*;*

(iii) $\mathcal{J}(\Phi^I(v)) \subseteq \mathcal{J}(v)$*;*

(iv) if $v \in \mathcal{S}^{**J}$*, then* $\Phi^I(v) \in \mathcal{S}^{**}$.

Parts (i) and (ii) of Corollary 2 imply that $\mathcal{S}^{**I}$ is precisely the set of fixed points of Φ^I. Combining part (i) with the counterpart of (iv) for side J shows that a single application of Φ^I followed by a single application of Φ^J takes any stable payoff into $\mathcal{S}^{**}$.

Theorem 3 (Existence)**.** *For every* $v \in \mathcal{S}$*, we have* $\Phi^J(\Phi^I(v)) \in \mathcal{S}^{**}$*. In particular,* $\mathcal{S}^{**} \neq \emptyset$*.*

4.4. **Characterization of Φ^I and the Lattice Structure of $\mathcal{S}^{**}$.** For $v \in \mathcal{S}$, let

$$\mathcal{D}^I(v) = \{w \in \mathcal{S}^{**I} \mid w \geq_I v\}.$$

By Corollary 2, $\Phi^I(v) \in \mathcal{D}^I(v)$, so $\mathcal{D}^I(v) \neq \emptyset$ for every $v \in \mathcal{S}$. Moreover, every $w \in \mathcal{D}^I(v)$ belongs to $\mathcal{S}^{**I}$ and is thus a fixed point of Φ^I by Corollary 2.(ii). Since $w \geq_I v$ and Φ^I is isotone, $w = \Phi^I(w) \geq_I \Phi^I(v)$. These observations lead to the following characterization of Φ^I, which is the side-level counterpart of the player-level characterization of $F^i(v)$ in Theorem 1.(iii).

Theorem 4 (Characterization of Φ^I)**.** *For* $v \in \mathcal{S}$*,*

$$\Phi^I(v) = \bigwedge\nolimits^I \mathcal{D}^I(v).$$

Thus, $\Phi^I(v)$ is the *lowest stable payoff to which no player in* I *has a legitimate objection and that* $\geq_I$*-dominates* v. This is the sense in which the tâtonnement process that iterates F^I clears the objections of players in I with *minimal payoff increments*. The characterization also implies that Φ^I is *independent of the ordering* $(i_1, \ldots, i_m)$ used to define it.

We turn to the lattice structure of $\mathcal{S}^{**}$. Recall that for every nonempty $X \subseteq \mathcal{S}$, both $\bigwedge^I X$ and $\bigvee^I X$ belong to $\mathcal{S}$. The next result proves that the conclusion regarding the operator $\bigwedge^I$ extends to $\mathcal{S}^{**I}$.

Lemma 9 ($\mathcal{S}^{**I}$ Is Closed under $\bigwedge^I$)**.** *For any* $X \subseteq \mathcal{S}^{**I}$ *with* $X \neq \emptyset$*, we have* $\bigwedge^I X \in \mathcal{S}^{**I}$*.*

We can now establish that $(\mathcal{S}^{**}, \geq_I)$ is a complete lattice, with meet and join recovered from the operators Φ^I and Φ^J.

Theorem 5 ($\mathcal{S}^{**}$ Is a Complete Lattice)**.** $(\mathcal{S}^{**}, \geq_I)$ *is a complete lattice, and for any* $X \subseteq \mathcal{S}^{**}$ *with* $X \neq \emptyset$*,*

$$\bigvee\nolimits^I_{\mathcal{S}^{**}} X = \Phi^I(\bigvee\nolimits^I X), \qquad \bigwedge\nolimits^I_{\mathcal{S}^{**}} X = \Phi^J(\bigvee\nolimits^J X).$$

Furthermore, the $\geq_I$*-greatest element of* $\mathcal{S}^{**}$ *is* $\Phi^J(\bar{v}^I)$ *and the* $\geq_I$*-least element is* $\Phi^I(\bar{v}^J)$*.*

To prove the result, consider a nonempty set $X \subseteq \mathcal{S}^{**}$. By Lemma 9 applied to the J side, $\bigvee^I X = \bigwedge^J X \in \mathcal{S}^{**J}$, so Corollary 2.(iv) yields $\Phi^I(\bigvee^I X) \in \mathcal{S}^{**}$. By Corollary 2.(ii), $\Phi^I(\bigvee^I X) \geq_I \bigvee^I X \geq_I x$ for all $x \in X$, so $\Phi^I(\bigvee^I X)$ is a $\geq_I$-upper bound for X in $\mathcal{S}^{**}$. For any $\geq_I$-upper bound $w \in \mathcal{S}^{**}$ of X, we have $w \geq_I \bigvee^I X$ and $w \in \mathcal{S}^{**I}$, so $w \in \mathcal{D}^I(\bigvee^I X)$, and Theorem 4 implies that $w \geq_I \Phi^I(\bigvee^I X)$. Thus, $\Phi^I(\bigvee^I X)$ is the $\geq_I$-least upper bound of X in $\mathcal{S}^{**}$. The meet formula follows symmetrically.

Finally, $\bar{v}^I \in \mathcal{S}^{**I}$ by Corollary 1. Since $\mathcal{S}^{**} \subseteq \mathcal{S}^{**J} \subseteq \mathcal{D}^J(\bar{v}^I)$, Corollary 2.(iv) and Theorem 4, both applied to the J side, imply that $\Phi^J(\bar{v}^I)$ is the $\geq_J$-least, and hence the $\geq_I$-greatest, element of $\mathcal{S}^{**}$. The argument for the $\geq_I$-least element $\Phi^I(\bar{v}^J)$ is symmetric. Thus, each side-optimal payoff in $(\mathcal{S}^{**}, \geq_I)$ is reached by a one-sided procedure.

4.5. **Fixed-Point Characterizations of $\mathcal{S}^{**}$.** Unlike $\mathcal{S}^R$ and $\mathcal{S}^*$, which belong to Moldovanu's (1993) class of solutions based only on matched players' direct outside options, $\mathcal{S}^{**}$ does not lend itself to an immediate fixed-point formulation. Payoff adjustments can activate or disrupt payoff-preserving rematching chains, changing the single-match games relevant to a player's objection and creating a moving target for clearing it. Our construction overcomes this difficulty: both compositions $\Phi^J \circ \Phi^I$ and $F^J \circ F^I$ yield fixed-point characterizations of $\mathcal{S}^{**}$.

Theorem 6 (Fixed-Point Characterizations)**.** *$\mathcal{S}^{**}$ is the set of fixed points of $\Phi^J \circ \Phi^I$. Moreover, $\mathcal{S}^{**}$ is also the set of fixed points of $F^J \circ F^I$.*

To prove the first statement, note that if v is a fixed point of $\Phi^J \circ \Phi^I$, then Theorem 3 yields $v = \Phi^J(\Phi^I(v)) \in \mathcal{S}^{**}$. Conversely, $\Phi^J(v) = \Phi^I(v) = v$ for every $v \in \mathcal{S}^{**}$ by Corollary 2.(ii) applied to both sides. Hence, $\Phi^J \circ \Phi^I$ projects $\mathcal{S}$ onto $\mathcal{S}^{**}$.

The second statement requires a separate argument because $F^J \circ F^I$ need not map $\mathcal{S}$ into $\mathcal{S}^{**}$. The Appendix establishes directly that no player has a legitimate objection at a fixed point of this composition.

Either fixed-point characterization offers an alternative route to the lattice structure of $\mathcal{S}^{**}$ via Tarski's (1955) fixed-point theorem, which establishes that the fixed points of an isotone self-map of a complete lattice form a nonempty complete lattice. The single-player operators are isotone by Theorem 1 and its symmetric counterpart, so the compositions $F^J \circ F^I$ and $\Phi^J \circ \Phi^I$ are isotone self-maps of the complete lattice $(\mathcal{S}, \geq_I)$. The route via $F^J \circ F^I$ establishes the nonemptiness and lattice structure of $\mathcal{S}^{**}$ without relying on finite termination of the tâtonnement process.

The constructive approach to Theorem 5 has several advantages over the high-level routes provided by Tarski's theorem. First, $\Phi^J \circ \Phi^I$ maps every stable payoff into $\mathcal{S}^{**}$, so existence follows directly, and by Theorem 2 the process defining Φ^I and Φ^J yields an exact algorithm that reaches $\mathcal{S}^{**}$ from any stable payoff in a uniformly bounded number of steps. Second, the operators Φ^I and Φ^J provide explicit formulae for the meet and join in $(\mathcal{S}^{**}, \geq_I)$ and identify its lattice extremes. Third, each operator clears objections on its side with minimal payoff increases for players on that side, independently of the ordering of single-player operators.

5. Other Structural Properties of $\mathcal{S}^{**}$

In this section, we further examine the structure of the refined solution. We show that $\mathcal{S}^{**}$ need not be a sublattice of the ambient lattice $(\mathcal{S}, \geq_I)$, nor a convex subset of $\mathbb{R}^N$. We also provide a geometric description of $\mathcal{S}^{**}$.

5.1. **Failure of the Sublattice Property and of Convexity of $\mathcal{S}^{**}$.** As reviewed in Section 2, $(\mathcal{S}, \geq_I)$ is a complete lattice whose join $\bigvee_{\mathcal{S}}^I$ and meet $\bigwedge_{\mathcal{S}}^I$ coincide with the componentwise operators $\bigvee^I$ and $\bigwedge^I$ on the set of feasible payoffs $\mathcal{F}$. We show that $\mathcal{S}^*$ and $\mathcal{S}^{**}$ need not be closed under the componentwise operators $\bigvee^I$ and $\bigwedge^I$, and hence need not be sublattices of $(\mathcal{S}, \geq_I)$. Furthermore, while $\mathcal{S}$ is a convex polytope in $\mathbb{R}^N$, neither refinement is necessarily a convex subset of $\mathbb{R}^N$.

Theorem 7. *$\mathcal{S}^*$ and $\mathcal{S}^{**}$ are not necessarily sublattices of $(\mathcal{S}, \geq_I)$ and not necessarily convex subsets of $\mathbb{R}^N$.*

Although $(\mathcal{S}^{**}, \geq_I)$ is itself a complete lattice, it need not be a sublattice of $(\mathcal{S}, \geq_I)$, and neither need $(\mathcal{S}^*, \geq_I)$: the join $\bigvee_{\mathcal{S}}^I\{v, w\}$ of a pair of payoffs $v, w \in \mathcal{S}^{**}$ evaluated in $\mathcal{S}$ can fall outside $\mathcal{S}^*$, and hence outside $\mathcal{S}^{**}$. Figure 4 exhibits such a pair in a 5×5 assignment game with efficient matching $\eta(i) = i + 5$ ($I = \{1, \ldots, 5\}$, $J = \{6, \ldots, 10\}$), and two stable payoffs v and w represented next to the corresponding nodes in the first and second panel, respectively. A link between two players indicates a binding constraint: horizontal links represent efficiently matched pairs under both v and w, and diagonal links represent binding stability constraints under the corresponding stable payoff profile. Thus, for any link ij shown in the v panel we have $a_{ij} = v_i + v_j$, whereas for any link ij shown in the w panel we have $a_{ij} = w_i + w_j$. For every pair (i, j) linked in neither panel, choose $a_{ij} < 20$. Such values can be selected without violating our genericity assumption that no two matchings generate the same total value (an assumption that in particular requires every a_{ij} to be non-zero).

Both v and w belong to $\mathcal{S}^{**}$, but their join $\bigvee_{\mathcal{S}}^I\{v, w\}$ does not belong to $\mathcal{S}^*$: player 10 still receives more than $a_{5,10}/2$ under the join, yet—unlike under v and w—she has no binding outside option. Taking the componentwise maximum on side I introduces slack in the stability constraints for both pairs $(4, 10)$ and $(2, 10)$, which support player 10's more-than-equal share in her match with player 5 under v and w, respectively. Player 10's excessive payoff under the join is thus unjustified, so $\bigvee_{\mathcal{S}}^I\{v, w\} \notin \mathcal{S}^*$. The same example with the roles of I and J interchanged shows that $\mathcal{S}^*$ and $\mathcal{S}^{**}$ need not be closed under $\bigwedge_{\mathcal{S}}^I$ either.

The example in Figure 4 also shows that, unlike the convex polytope $\mathcal{S}$, neither $\mathcal{S}^*$ nor $\mathcal{S}^{**}$ is convex: the average $(v + w)/2$ of these two solutions lies outside $\mathcal{S}^*$. Player 10's more-than-equal share in her match with player 5 is justified by a binding outside option—via the match $(4, 10)$ under v and via the match $(2, 10)$ under w. Both the join and the average make the two supporting stability constraints slack, so player 10's more-than-equal payoff is no longer justified.

5.2. **Geometric Structure of $\mathcal{S}^{**}$.** We now describe the geometric structure of $\mathcal{S}^{**}$. Let $\mathfrak{R} = \{\mathcal{R}(v) \mid v \in \mathcal{S}\}$, and for $R \in \mathfrak{R}$, let $\mathcal{S}(R)$ denote the set of stable payoffs v for which every pair

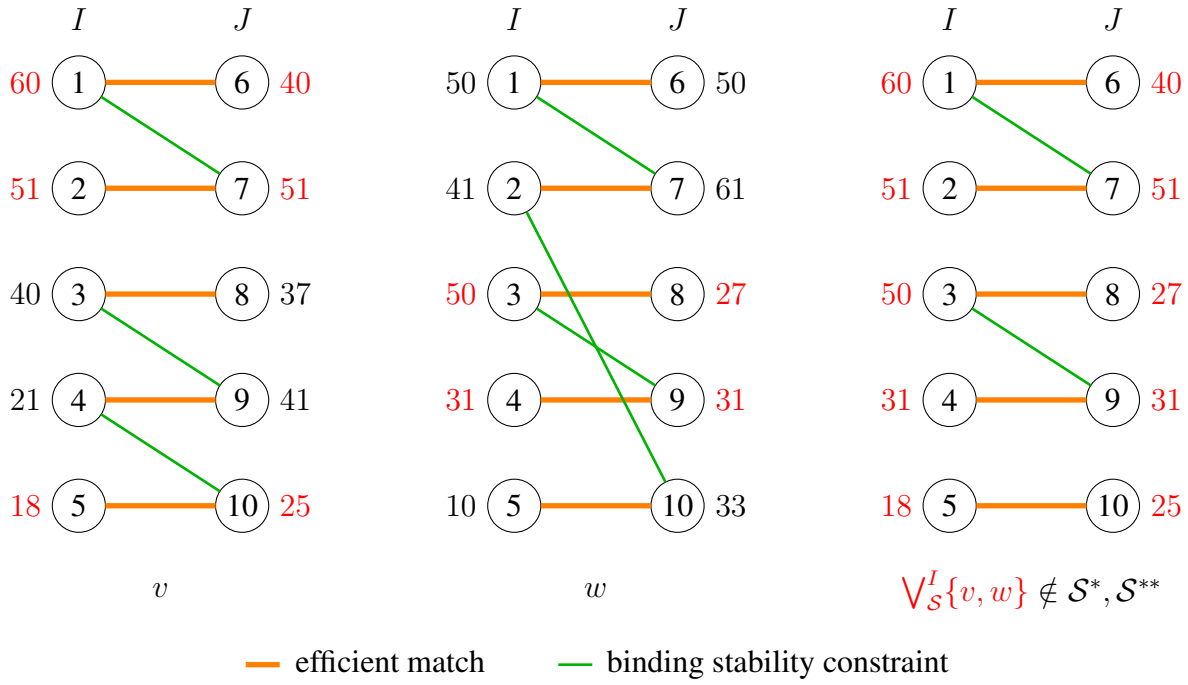

FIGURE 4. $\mathcal{S}^*$ and $\mathcal{S}^{**}$ are not sublattices of $\mathcal{S}$: $v, w \in \mathcal{S}^{**}$, while $\bigvee_{\mathcal{S}}^{I}\{v,w\} \notin \mathcal{S}^*$.

in R is residual, that is, $R \subseteq \mathcal{R}(v)$. By Lemma 3,

$$\mathcal{S}(R) = \bigcap_{ij \in R} \left(\bigcup_{\substack{\text{distinct } i_0,\ldots,i_{\bar{s}} \in I^* \\ i_0 = i,\ i_{\bar{s}} = \eta(j)}} \bigcap_{s=0}^{\bar{s}-1} \{ v \in \mathcal{S} \mid v_{i_{s+1}} + v_{\eta(i_s)} = a_{i_{s+1}\eta(i_s)} \} \right). \tag{6}$$

It is easy to see that

$$\mathcal{S}^{**I} = \bigcup_{R \in \mathfrak{R}} \left(\bigcap_{i \in I^*} \left(\bigcup_{j:\, ij \in R} \{ v \in \mathcal{S}(R) \mid v_i \geq \omega_i(\Gamma_{ij}) \} \right) \right), \tag{7}$$

with the symmetric expression for $\mathcal{S}^{**J}$. Geometrically, $\mathcal{S}$ is a polytope in $\mathbb{R}^N$; each $\mathcal{S}(R)$ is a finite union of intersections of faces of $\mathcal{S}$ (after distributing the unions in (6) over the intersections), and each set $\{v \in \mathcal{S}(R) \mid v_i \geq \omega_i(\Gamma_{ij})\}$ is obtained by further intersecting with the half-space $v_i \geq \omega_i(\Gamma_{ij})$. Thus, (6) and (7) express $\mathcal{S}^{**I}$ and $\mathcal{S}^{**J}$—and consequently $\mathcal{S}^{**} = \mathcal{S}^{**I} \cap \mathcal{S}^{**J}$—as finite unions of closed polytopes, which also establishes compactness of the solution.

6. IMPLEMENTATION

We seek support for $\mathcal{S}^{**}$ from the Nash (1953) Program: does our cooperative solution conform with non-cooperative equilibria (possibly refined as in Nash's demand game) of some sensible mechanism? If so, we have more confidence in the cooperative solution and conversely the non-cooperative game is legitimized by the agreement.

We propose a mechanism consisting of a demand phase potentially followed by a bargaining phase. Players participate in demand rounds $t = 0, 1, \ldots$ until either they reach a single-match game or no more matches are possible. If a single-match game is reached at the end of round t, then the remaining players bargain in rounds $t' \geq t + 1$.[11] There is a discount factor $\delta < 1$ between any consecutive rounds: payoffs realized in round t are discounted by δ^t. The transition to bargaining entails an additional friction, which we model as a (one-time) discount factor $\beta < 1$ applied to the payoffs of players who proceed to the bargaining phase. For instance, this could reflect a cost of decentralized agreements relative to the centralized matching in the demand phase. Players bargain according to one of the many protocols known to generate a unique MPE outcome that conforms with the OOP as $\delta \to 1$.

The demand phase is a multilateral version of the Nash (1953) demand game in the spirit of Kamecke (1989) and Pérez-Castrillo and Sotomayor (2002). In those mechanisms, each player submits a single demand, and pairs with compatible demands are matched according to carefully designed protocols. Since the matching process accounts for outside options, stability emerges as a natural prediction. The set of equilibria thus traces out all of $\mathcal{S}$. Variations select the side-optimal payoffs of the core: for instance, the I-optimal payoff is achieved by granting players in I a first-mover advantage.

Nash-style demand games are notorious for their multiplicity of equilibria (for instance, all divisions in a trivial two-player assignment game). What distinguishes our mechanism, and what sharpens the prediction from $\mathcal{S}$ to $\mathcal{S}^{**}$, is the bargaining endgame. When the submitted demands lead to a single-match subgame, the remaining players bargain, and the outcome conforms with the OOP. A player who contemplates a more aggressive demand must then weigh the prospect of reaching such a subgame, where she obtains her OOP payoff—equal split when no outside option binds, and the division induced by a binding outside option otherwise. The bargaining phase is critical in implementing the equity subject to no objection principle on which $\mathcal{S}^{**}$ is based.

In every demand round in which the set of remaining players is N', each player $k \in N'$ submits a demand $v_k \geq 0$. The gap between $i \in I \cap N'$ and $j \in J \cap N'$ is defined by $a_{ij} - v_i - v_j$. Matches in the round proceed in multiple steps. First, if the maximum gap is non-negative, a matching that maximizes the number of pairs attaining the maximum gap is implemented. When multiple such matchings exist, the procedure favors matching players who are efficiently matched in the remaining subgame. The procedure is then repeated within the round for the residual players as long as their maximum gap remains non-negative.

More precisely, let $N^t \subseteq N$ denote the set of remaining players at the beginning of round t, with $N^0 = N$. If $t \geq 1$ and N^t is a single-match game, then players bargain in round t according to the chosen bargaining protocol. Otherwise, players engage in a demand game in round t, in which each $k \in N^t$ simultaneously submits a demand $v_k^t \geq 0$. Matching in round t proceeds in a sequence of steps determined by the submitted demands. Set $N_0^t = N^t$, and for each *step* $r \geq 0$,

[11] It is convenient for our definitions to assume that a demand round takes place at $t = 0$ even if N is a single-match game.

if N_r^t contains only players on one side of the market, then all matches under $\mu_0^t, \ldots, \mu_{r-1}^t$ (this set is empty if $r = 0$) are implemented in round t, with payoffs as specified below, and the game ends. Otherwise, recursively compute the greatest gap g_r^t among pairs of players in N_r^t given their demands:

$$g_r^t = \max_{(i,j)\in(I\cap N_r^t)\times(J\cap N_r^t)} (a_{ij} - v_i^t - v_j^t).$$

If $g_r^t \geq 0$, let μ_r^t be a maximum-cardinality matching among the pairs in N_r^t with gap g_r^t, chosen, if several exist, to *match the most players who are efficiently matched in the assignment game restricted to* N_r^t. Let N_{r+1}^t denote the set of players in N_r^t not matched under μ_r^t. We then proceed to step $r+1$ in round t following the same procedure. If $g_r^t < 0$, then all matches under $\mu_0^t, \ldots, \mu_{r-1}^t$ (this set is empty if $r = 0$) are implemented in round t, and the corresponding players exit the game, with each player k matched under $\mu_{r'}^t$ receiving a payoff of $v_k^t + g_{r'}^t/2$, for $r' = 0, \ldots, r-1$. The game proceeds to round $t+1$ with the remaining set of players $N^{t+1} = N_r^t$.

The matching procedure depends only on the match values and the submitted demands, so it can be implemented in every subgame for any profile of demands. To make the selection among matchings consistent with these rules part of the equilibrium, in the spirit of Simon and Zame (1990), we introduce a *market maker* to the game who implements matches and obtains a positive payoff for following the rules above and zero otherwise. In the first step of the first demand round, the market maker breaks ties in favor of the players in N^*. At later steps and in later rounds, the set of efficiently matched players prioritized by the matching procedure is well defined because our genericity assumption is inherited in every subgame (see footnote 6), and it need not consist of players in N^*.

For this non-cooperative game, we impose the following refinements on subgame perfect equilibria:

(1) *No delay*: all matches happen in the first demand round, and no further match is possible (that is, all remaining players, if any, are on the same side of the market).
(2) *Markov bargaining*: players use Markov strategies in the bargaining phase.

In the bargaining phase, when a single-match subgame Γ is reached, then play proceeds according to some MPE of Γ. For every discount factor δ, all MPEs of Γ are payoff equivalent, so each player $k \in \Gamma$ expects the same payoff $\omega_k^\delta(\Gamma)$ regardless of the history of play in the demand phase.

The first result of this section establishes one direction of the Nash Program correspondence: refined equilibria of the demand-and-bargaining game select payoffs in $\mathcal{S}^{**}$ as frictions vanish.

Theorem 8. *For any sequence $(\delta, \beta) \to (1, 1)$ and any associated sequence of subgame perfect equilibria satisfying the two refinements, every limit point of the sequence of equilibrium payoffs belongs to $\mathcal{S}^{**}$.*

The proof relies on the following two lemmas.

Lemma 10 (No Money Left on the Table in Demand Rounds)**.** *In every equilibrium, there cannot be a positive gap in any demand round.*

Lemma 11 (No Delay Implies Stability)**.** *Every equilibrium that satisfies the no-delay refinement implements the efficient matching η and yields a stable payoff vector.*

Lemmas 10 and 11 reduce the proof of Theorem 8 to showing that limit payoffs admit no legitimate objections. The argument is as follows. Suppose player $i \in I^*$ increases her demand above the equilibrium level. Player i is then left unmatched in the first round, while the market maker, breaking ties in favor of players in N^*, rematches the remaining players along a rematching chain from i to $\eta(j)$ for some $j \in J^*$. Either i and j proceed to bargain, together with the players in U, in the single-match game Γ_{ij}, or j matches with a player $u \in U$ and i is left without any potential partner. The binding constraints along the chain imply that ij is a residual pair at the limit payoff $\hat{v}$. In the first case, the deviation is unprofitable only if i's payoff is at least her bargaining payoff in Γ_{ij}, discounted by β and δ. This bargaining payoff converges to $\omega_i(\Gamma_{ij})$ as $(\delta, \beta) \to (1, 1)$, so $\hat{v}_i \geq \omega_i(\Gamma_{ij})$. In the second case, the match between j and u implies $\hat{v}_j \leq a_{uj}$, and Lemma 4.(ii) then leads to $\hat{v}_i \geq \omega_i(\Gamma_{ij})$. Thus, there is a residual pair $ij \in \mathcal{R}(\hat{v})$ with $\hat{v}_i \geq \omega_i(\Gamma_{ij})$, which means that i's objection to $\hat{v}$ is not legitimate.

We next establish a converse of Theorem 8: every point $v \in \mathcal{S}^{**}$ assigning positive payoffs to players in N^* can be implemented in a refined equilibrium of our non-cooperative game as frictions vanish. The hypothesis that $v_k > 0$ for every player $k \in N^*$ sustains the demand profile v against upward deviations. When a player k deviates to a larger demand, the maximum-cardinality requirement and the tie-breaking in favor of players in N^* allow two outcomes. Either a player on the opposite side is matched with some $u \in U$, leaving k without any potential partner, or the remaining players are rematched according to a kl-reassignment under v for some player l on the opposite side. In the latter case, the market maker selects l with $v_k \geq \omega_k(\Gamma_{kl})$ in the equilibrium we construct, as the rules permit. Such an l exists because k has no legitimate objection at $v \in \mathcal{S}^{**}$. The deviation then yields k zero in the first case and at most $\beta\delta\omega_k^\delta(\Gamma_{kl})$ in the second. Since $\omega_k^\delta(\Gamma_{kl}) \to \omega_k(\Gamma_{kl})$ as $\delta \to 1$ and $\omega_k(\Gamma_{kl}) \leq v_k$, with $v_k > 0$ and $\beta < 1$, the deviation is not profitable for δ sufficiently close to 1. If we had $v_k = 0$, the potentially positive bargaining payoff $\omega_k^\delta(\Gamma_{kl})$ for $\delta < 1$ would make the deviation profitable, and the construction would break down.

The positivity hypothesis is not redundant—a player matched under the efficient matching η may nonetheless get zero at some payoff profiles in $\mathcal{S}^{**}$, as the example from Figure 5 illustrates. The next lemma delineates the scope of the hypothesis.

Lemma 12 (Positive Solutions)**.** *The elements of $\mathcal{S}^{**}$ that assign positive payoffs to all players in N^* form a nonempty dense subset of $\mathcal{S}^{**}$. Furthermore, if $U = \emptyset$, or every player in U is a less valuable partner than every player in N^* on the same side of the market as U, then every element of $\mathcal{S}^{**}$ assigns positive payoffs to all players in N^*.*

The lemma shows that the next theorem is never vacuous. In every assignment game satisfying our genericity assumption, the subset of $\mathcal{S}^{**}$ to which it applies is dense in $\mathcal{S}^{**}$. Every refined payoff can therefore be approximated arbitrarily closely by payoffs to which the theorem applies.

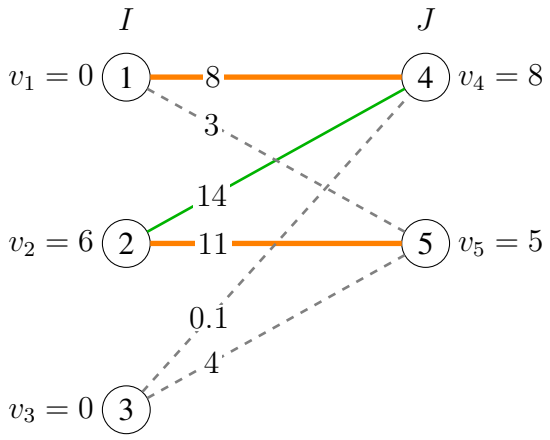

FIGURE 5. An assignment game with $v \in \mathcal{S}^{**}$ in which $v_1 = 0$ and $1 \notin U$

Moreover, whenever players in U are less valuable partners than the players in N^*—and, in particular, whenever $U = \emptyset$—the positivity hypothesis of the theorem holds over the entire refined solution $\mathcal{S}^{**}$.

Theorem 9. *Fix $v \in \mathcal{S}^{**}$ with $v_k > 0$ for all $k \in N^*$. For any $\beta < 1$, there exists $\underline{\delta} < 1$ such that for all $\delta > \underline{\delta}$, the game with frictions (δ, β) admits a subgame perfect equilibrium satisfying the two refinements in which the efficient matching η forms in the first demand round and players obtain payoffs v.*

7. CONCLUSION

We have proposed a refinement of the core of the assignment game of Shapley and Shubik (1971) that incorporates bargaining possibilities Shapley and Shubik themselves identified as missing from the formulation of the core. The refinement is motivated from both a cooperative and a non-cooperative perspective, in keeping with the precepts of the Nash (1953) Program. In single-match assignment games, standard bargaining protocols support the outside option principle: outside options provided by the long side are relevant only if they are binding relative to equal division of the efficient match's value. From the cooperative perspective, an objection by a player to a stable payoff leads to a payoff-preserving rematching in which all efficiently matched players except the objector and one player on the opposite side are matched among themselves at their proposed payoffs; the two exempted players, along with the players unmatched under the efficient matching, are left in a single-match game whose outcome conforms to the outside option principle. The objection is legitimate if every such rematching leaves the objector with a strictly higher payoff than her proposed one. Our refinement $\mathcal{S}^{**}$ consists of the stable payoffs to which no player has a legitimate objection.

Our cooperative construction recasts the no-blocking constraints among matched players as bounds on same-side payoff differences, with constants given by differences in coalitional worths.

These bounds identify when residual partners become available to an objecting player i as her payoff increases. Her OOP payoffs with these partners determine the smallest stability-preserving increase in her side's payoffs that clears her objection, defining the operator F^i.

Cycling the operators F^i over one side of the market takes any stable payoff, in a number of steps bounded uniformly over the core, to a "nearest" stable payoff at which no player on that side has a legitimate objection. At every step the set of players with legitimate objections on the other side weakly shrinks. The one-sided procedures define operators Φ^I and Φ^J on $\mathcal{S}$, and applying them in succession clears the objections of both sides. The solution $\mathcal{S}^{**}$ is nonempty, equals the set of fixed points of $\Phi^J \circ \Phi^I$, and forms a complete lattice—a structure intrinsic to the refinement rather than inherited from the core, since $\mathcal{S}^{**}$ is in general not a sublattice of $\mathcal{S}$. The meet, join and lattice extremes of $\mathcal{S}^{**}$ are given by explicit formulae in terms of Φ^I and Φ^J.

We anchor the refinement in the Nash (1953) Program via a demand-and-bargaining game: as frictions vanish, limit payoffs of refined subgame perfect equilibria belong to $\mathcal{S}^{**}$, and conversely every payoff in $\mathcal{S}^{**}$ that is positive for all efficiently matched players arises in this fashion. The threat of relegation to bilateral bargaining upon an aggressive demand is the non-cooperative counterpart of the equity subject to no objection principle that defines $\mathcal{S}^{**}$. The cooperative and non-cooperative analyses thus reinforce each other.

Throughout, we have maintained the assumption of generic match values. It would be useful to examine how the refinement behaves when multiple efficient matchings coexist. Another natural avenue for future research is to extend the analysis to many-to-one matching.

In Shapley and Shubik's words, "the core gives no consideration to the ability of individuals or coalitions to *obstruct* outcomes, though this is often a powerful lever in bargaining." Our refinement is the set of stable payoffs that no individual can credibly obstruct.

APPENDIX

Proof of Lemma 2. Toward a contradiction, consider a payoff profile v in $\mathcal{S}^{**} \setminus \mathcal{S}^{*}$. For such a v, there exists an efficient match (i, j) such that either $v_j > a_{ij}/2$ and

$$v_{i'} + v_j > a_{i'j}, \quad \forall i' \in I \setminus \{i\} \tag{8}$$

or the symmetric conditions hold with the roles of the two market sides interchanged. Without loss of generality, assume we are in the former case. It follows that the only reassignment under v that involves i is an ij-reassignment. As $v \in \mathcal{S}^{**}$, this yields $v_i \geq \omega_i(\Gamma_{ij})$.

Since $j = \eta(i)$, all matches in Γ_{ij} create value at most a_{ij}. It follows that $\omega_i(\Gamma_{ij}) + \omega_j(\Gamma_{ij}) = a_{ij}$. Since $v_i + v_j = a_{ij}$, it must be that $v_j \leq \omega_j(\Gamma_{ij})$. Therefore, $a_{ij}/2 < v_j \leq \omega_j(\Gamma_{ij})$.

The inequality $a_{ij}/2 < \omega_j(\Gamma_{ij})$ is possible only if $U \subset I$ and $\omega_j(\Gamma_{ij}) = a_{uj}$ for some $u \in U$. Since v is stable, we have that $v_u = 0$ and $v_j \geq a_{uj}$. It follows that $v_j = a_{uj}$, which means that $v_j + v_u = a_{uj}$, contradicting (8). □

Proof of Lemma 3. To prove the "only if" part, suppose that $ij \in \mathcal{R}(v)$. Let (v', η') be an ij-reassignment under v. Set $i_0 = i$, $j_0 = \eta(i_0)$, and as long as $j_s \neq j$, define recursively

$$i_{s+1} = \eta'(j_s) \text{ and } j_{s+1} = \eta(i_{s+1}) \text{ for } s \geq 0.$$

The sequence ends at a step $\bar{s}$ if $j_{\bar{s}} = j$; set $\bar{s} = \infty$ if this situation is not reached. Since η matches every player in I^* to a player in J^*, and η' matches every player in $I^* \setminus \{i\}$ to a player in $J^* \setminus \{j\}$, it follows by induction that for every $s = 1, \ldots, \bar{s}$, we have that $i_s \in I^* \setminus \{i\}$ and $j_s \in J^*$, with $j_s \in J^* \setminus \{j\}$ for $s < \bar{s}$. In particular, the sequence is well defined ($i_s, j_s \neq \emptyset$ for $s \leq \bar{s}$).

Another inductive argument now shows that for every $s = 0, \ldots, \bar{s}$, all players $i_0, j_0, \ldots, i_s, j_s$ are distinct. To prove the induction step of this claim, note that by the inductive hypothesis all terms in the sequence $j_0, \ldots, j_{s-1}$ are distinct, and η' induces a one-to-one matching from $j_0, \ldots, j_{s-1}$ to $i_1, \ldots, i_s$, so all terms in the latter sequence must be distinct. We argued that $i_1, \ldots, i_s \in I^* \setminus \{i\}$, so all terms of the sequence $i_0, \ldots, i_s$ are distinct. Since η induces a one-to-one matching from $i_0, \ldots, i_s$ to $j_0, \ldots, j_s$, all terms in the latter sequence are also distinct.

Since I^* is finite, it must be that the sequence ends in a finite number of steps $\bar{s}$, and $j_{\bar{s}} = j$. The fact that (v', η') is an ij-reassignment under v implies that

$$v_{i_{s+1}} + v_{j_s} = a_{i_{s+1} j_s} \text{ for } s = 0, \ldots, \bar{s} - 1.$$

It follows that $i_0 = i, \ldots, i_{\bar{s}} = \eta(j)$ defines a rematching chain under v.

To prove the "if" part, consider a rematching chain $(i_s)_{s=0}^{\bar{s}}$ under v contained in I^* with $i_0 = i$ and $i_{\bar{s}} = \eta(j)$. Define the matching η' such that: (1) η' matches i_{s+1} with $\eta(i_s)$ for $s = 0, 1, \ldots, \bar{s} - 1$; (2) $\eta'(k) = \eta(k)$ for $k \in N^* \setminus (\{i_0, \ldots, i_{\bar{s}}\} \cup \{\eta(i_0), \ldots, \eta(i_{\bar{s}})\})$; (3) η' conforms with the OOP in the single-match game Γ_{ij}. Let v' be given by (1) $v'_k = v_k$ for all $k \in N^* \setminus \{i, j\}$; (2) $v'_k = \omega_k(\Gamma_{ij})$ for all players k in the single-match game Γ_{ij}. Clearly, (v', η') is an ij-reassignment under v, so $ij \in \mathcal{R}(v)$.

For the final claim, the "only if" part of the first claim yields a rematching chain $i_0 = i, \ldots, i_{\bar{s}} = \eta(j)$ under v. If $i' \in \{i_0, \ldots, i_{\bar{s}}\}$, the truncated chain ending at i' is a rematching chain under v; otherwise, $v_{i'} + v_j = a_{i'j}$ and $j = \eta(i_{\bar{s}})$ imply that $i_0, \ldots, i_{\bar{s}}, i'$ is a rematching chain under v. In either case, the "if" part yields $i\eta(i') \in \mathcal{R}(v)$. □

Proof of Lemma 4. The argument preceding the lemma proves part (i).

To prove part (ii), assume that $U \cap I \neq \emptyset$. Then, $U \subset I$, so j is the only player in $\Gamma_{ij} \cap J$. Under the OOP, only the players in the unique match that generates maximum value in Γ_{ij} (see footnote 6) receive positive payoffs. Since $\omega_i(\Gamma_{ij}) > v_i \geq 0$, this match is (i, j). Hence, for all $u \in U$, we have $a_{ij} \geq a_{uj}$ and $\omega_j(\Gamma_{ij}) \geq a_{uj}$. Combined with part (i), this yields $v_j - a_{uj} \geq v_j - \omega_j(\Gamma_{ij}) \geq \omega_i(\Gamma_{ij}) - v_i$. □

Proof of Lemma 5. Fix $w \in \mathcal{F}$ and $i, i' \in I^*$. For any sequence $i = i_0, \ldots, i_{\bar{s}} = i'$ of distinct players in I^*, the pairs $(i_{s+1}, \eta(i_s))$ for $s < \bar{s}$ and $(k, \eta(k))$ for $k \in I^* \setminus \{i_0, \ldots, i_{\bar{s}}\}$ form a matching of $N^* \setminus \{i, \eta(i')\}$. Let T denote the total value of this matching. The match-value term

in (3) then equals $W(N^*) - a_{i'\eta(i')} - T$. Since $T \leq W(N^* \setminus \{i, \eta(i')\})$ for every sequence, part (i) follows if there is a sequence that induces the maximum-value matching of $N^* \setminus \{i, \eta(i')\}$.

Let μ be a maximum-value matching of $N^* \setminus \{i, \eta(i')\}$. As match values are positive, no two players left unmatched by μ lie on opposite sides of the market. Since $|I^* \setminus \{i\}| = |J^* \setminus \{\eta(i')\}|$, μ matches every player in $N^* \setminus \{i, \eta(i')\}$. View μ and η as sets of edges between I and J. The symmetric difference $\mu \triangle \eta$ contains no cycle. Indeed, such a cycle would exclude i and $\eta(i')$, and by the uniqueness of the efficient matching η, replacing the edges of μ along the cycle with those of η would strictly increase the value of μ. In $\mu \triangle \eta$, the players i and $\eta(i')$ have degree one and every other player has degree zero or two. Hence the edges of $\mu \triangle \eta$ form a single path from i to $\eta(i')$. The players in I^* along this path form a sequence $i = i_0, \ldots, i_{\bar{s}} = i'$ with $i_{s+1} = \mu(\eta(i_s))$, which induces the matching μ. This proves part (i).

For part (ii), let $w \in \mathcal{S}$. Since all stability slacks under w are non-negative, part (i) implies that $\tau_{ii'}(w) = 0$ if and only if some sequence $i = i_0, \ldots, i_{\bar{s}} = i'$ of distinct players in I^* has zero total slack. As each term in the associated sum is non-negative, this means that $\sigma_{i_{s+1}\eta(i_s)}(w) = 0$ for every $s < \bar{s}$, that is, the sequence is a rematching chain under w. By Lemma 3, such a chain exists if and only if $i\eta(i') \in \mathcal{R}(w)$.

For part (iii), write $i' := \eta(j)$. By part (ii), $ij \in \mathcal{R}(w)$ implies that $\tau_{ii'}(w) = 0$, that is, $w_i - w_{i'} = C_{ii'}$. Since $w \geq_I w'$ and $w_i = w'_i$, we have $w'_i - w'_{i'} \geq C_{ii'}$, that is, $\tau_{ii'}(w') \leq 0$. However, stability of w' and part (i) imply that $\tau_{ii'}(w') \geq 0$. Hence $\tau_{ii'}(w') = 0$, so $ij \in \mathcal{R}(w')$ by part (ii), and

$$w'_i - w'_{i'} = C_{ii'} = w_i - w_{i'}.$$

Since $w_i = w'_i$, we have $w'_{i'} = w_{i'}$, leading to $w'_j = w_j$ by feasibility. □

Proof of Lemma 6. Fix $w \in \mathcal{F}$, and note that $w_i - w_{i'} \leq C_{ii'}$ is equivalent to $\tau_{ii'}(w) \geq 0$ for all $i, i' \in I^*$. Since the pairs $(i, \eta(i))$ have zero slack by feasibility, no pair in $I^* \times J^*$ blocks w if and only if $\sigma_{i'\eta(i)}(w) \geq 0$ for all distinct $i, i' \in I^*$. If the latter condition holds, every sequence of distinct players in I^* has non-negative total slack, and Lemma 5.(i) implies that $\tau_{ii'}(w) \geq 0$ for all $i, i' \in I^*$ (with equality for $i = i'$).

Conversely, assume that $\tau_{ii'}(w) \geq 0$ for all $i, i' \in I^*$. For distinct $i, i' \in I^*$, the sequence $i_0 = i$, $i_1 = i'$ has total slack $\sigma_{i'\eta(i)}(w)$, and Lemma 5.(i) implies that $\sigma_{i'\eta(i)}(w) \geq \tau_{ii'}(w) \geq 0$. □

Proof of Lemma 7. For part (i), note that η restricted to $N^* \setminus \{i', \eta(i')\}$ is efficient there: adding the pair $(i', \eta(i'))$ to any matching of $N^* \setminus \{i', \eta(i')\}$ yields a matching of N^*, whose total value is at most $W(N^*)$. Hence, $W(N^*) - a_{i'\eta(i')} = W(N^* \setminus \{i', \eta(i')\})$, and substituting in the definition of $C_{ii'}$ yields the formula. For $i' = i$, the formula leads to $C_{ii} = 0$.

For part (ii), fix any $w \in \mathcal{S}$. Concatenating a sequence from i to i' of least total slack with a sequence from i' to i'' of least total slack, and deleting the loops created by repeated players, yields a sequence of distinct players from i to i''. Since slacks are non-negative at w, deleting a loop does not increase total slack, so by Lemma 5.(i) the total slack of the resulting sequence is at least

$\tau_{ii''}(w)$ and at most $\tau_{ii'}(w) + \tau_{i'i''}(w)$. Hence $\tau_{ii''}(w) \leq \tau_{ii'}(w) + \tau_{i'i''}(w)$, and substituting the definition of τ cancels the payoff terms, yielding $C_{ii''} \leq C_{ii'} + C_{i'i''}$. □

Proof of Lemma 8. Write $\tilde{w} := F^{i,d}(v)$, $\tau_{i'} := \tau_{ii'}(v)$ and $g_{i'} := \max\{d - \tau_{i'}, 0\}$ for $i' \in I^*$, so that $\tilde{w}_{i'} = v_{i'} + g_{i'}$, $\tilde{w}_{\eta(i')} = v_{\eta(i')} - g_{i'}$, and $\tilde{w}_u = 0$ for $u \in U$.

For part (i), $\tilde{w}$ is feasible, and $\tilde{w} \geq_I v$ implies that $\tilde{w}_{i'} \geq 0$ and $\tilde{w}_{i'} \geq a_{i'u}$ for $i' \in I^*$ and $u \in U \cap J$. By Lemma 6, no pair in $I^* \times J^*$ blocks $\tilde{w}$ if and only if

$$g_{i''} - g_{i'} \leq C_{i''i'} - v_{i''} + v_{i'} \quad \text{for all } i', i'' \in I^*. \tag{9}$$

Since $v \in \mathcal{S}$, Lemma 6 implies that the right side is non-negative. If $g_{i''} = 0$, the left side is at most zero, and hence does not exceed the right side. Otherwise $g_{i''} = d - \tau_{i''}$, and since $g_{i'} \geq d - \tau_{i'}$, the left side is at most $\tau_{i'} - \tau_{i''} = v_{i'} - v_{i''} + C_{ii'} - C_{ii''}$, which is at most $v_{i'} - v_{i''} + C_{i''i'}$ by the triangle inequality from Lemma 7.(ii). Hence, (9) holds. The stability conditions of Lemma 1 not yet verified are $\tilde{w}_j \geq 0$ and $\tilde{w}_j \geq a_{uj}$ for $j \in J^*$ and $u \in U \cap I$, which is the condition displayed in part (i).

For part (ii), let $w \in \mathcal{D}^{i,d}(v)$. By the lower bound derived before Definition 6, $w \geq_I \tilde{w}$, or equivalently, $\tilde{w} \geq_J w$. It follows that $\tilde{w}_j \geq w_j \geq \max\{0, \max_{u \in U \cap I} a_{uj}\}$ for all $j \in J^*$, so $\tilde{w} \in \mathcal{S}$ by part (i). Conversely, if $\tilde{w} \in \mathcal{S}$, then $\tilde{w} \in \mathcal{D}^{i,d}(v)$ by construction. Therefore, $\mathcal{D}^{i,d}(v) \neq \emptyset$ if and only if $\tilde{w} \in \mathcal{S}$, in which case $\tilde{w}$ is its $\geq_I$-least element, and thus its meet.

For part (iii), fix $i' \in I^*$. By Lemma 6, every $w \in \mathcal{S}$ with $w \geq_I v$ and $w_{i'} = v_{i'}$ satisfies

$$w_i - v_i \leq C_{ii'} - v_i + v_{i'} = \tau_{i'}.$$

Suppose that $\mathcal{D}^{i,\tau_{i'}}(v) \neq \emptyset$, and let $\hat{w} := F^{i,\tau_{i'}}(v)$. Part (ii) implies that $\hat{w} \in \mathcal{S}$. By construction, $\hat{w} \geq_I v$, $\hat{w}_i = v_i + \tau_{i'}$ and $\hat{w}_{i'} = v_{i'}$. Therefore, the bound is attained. □

Proof of Theorem 1. By (5), $F^i_i(v)$ is a minimum of maxima of non-decreasing functions of $(v_{i'})_{i' \in I^*}$ and of constants, and $F^i_{i'}(v) = \max\{v_{i'}, F^i_i(v) - C_{ii'}\}$ for $i' \in I^*$, so F^i is isotone.

Fix $v \in \mathcal{S}$ and $i \in I^*$, and write $\Delta := \Delta^i(v)$ and $\tau_{i'} := \tau_{ii'}(v)$ for $i' \in I^*$. By parts (i) and (ii) of Lemma 5, $\tau_{i'} \geq 0$, with equality if and only if $i\eta(i') \in \mathcal{R}(v)$. Since

$$\Delta = \min_{i' \in I^*} \max\{\tau_{i'},\ \theta_{ii'} - v_i\},$$

we have that $\Delta \geq 0$, with equality if and only if some $i' \in I^*$ satisfies $\tau_{i'} = 0$ and $v_i \geq \theta_{ii'}$, or equivalently $i\eta(i') \in \mathcal{R}(v)$ and $v_i \geq \theta_{ii'}$. The latter condition is equivalent to $i \notin \mathcal{I}(v)$. Therefore, $\Delta = 0$ if and only if $i \notin \mathcal{I}(v)$.

Let $I^+ := \{i' \in I^* \mid \tau_{i'} < \Delta\}$ denote the set of players in I whose payoffs strictly increase under $F^i(v)$; the payoffs of their efficient partners decrease by the same amounts.

Part (i). By Lemma 8.(i), $F^i(v) \in \mathcal{S}$ provided that $F^i_j(v) \geq \max\{0, \max_{u \in U \cap I} a_{uj}\}$ for all $j \in J^*$. If $\eta(j) \notin I^+$, then $F^i_j(v) = v_j$ and the bound holds by stability of v. Otherwise, write $i' := \eta(j) \in I^+$. Since $\Delta \leq \max\{\tau_{i'}, \theta_{ii'} - v_i\}$ and $\tau_{i'} < \Delta$, we have $\theta_{ii'} - v_i \geq \Delta > \tau_{i'} \geq 0$. In

particular, $\omega_i(\Gamma_{ij}) = \theta_{ii'} > v_i$, so Lemma 4 applies to the pair (i, j), and its parts (i) and (ii) yield

$$F^i_j(v) = v_j - (\Delta - \tau_{i'}) \geq v_j - (\theta_{ii'} - v_i) \geq \omega_j(\Gamma_{ij}) \geq \max\Big\{0, \max_{u \in U \cap I} a_{uj}\Big\}. \tag{10}$$

Part (ii). By construction, $F^i(v) \geq_I v$, with equality if and only if $\Delta = 0$. At the start of the proof we argued that $\Delta = 0$ is equivalent to $i \notin \mathcal{I}(v)$.

Part (iii). Let $i' \in I^*$ attain the minimum defining Δ. We have $\Delta \geq \tau_{i'}$ and $F^i_i(v) = v_i + \Delta \geq \theta_{ii'}$. By (4), $i\eta(i') \in \mathcal{R}(F^i(v))$, so $F^i(v) \in \mathcal{S}^{**i}$; part (ii) implies that $F^i(v) \in \{w \in \mathcal{S}^{**i} \mid w \geq_I v\}$. If $w \in \mathcal{S}^{**i}$ and $w \geq_I v$, then $F^i(w) = w$ by part (ii) and $F^i(w) \geq_I F^i(v)$ by isotonicity, so $w \geq_I F^i(v)$. Thus $F^i(v)$ is the $\geq_I$-least element of the set, and coincides with its meet.

Part (iv). Fix $j \in J^* \setminus \mathcal{J}(v)$ and write $i' := \eta(j)$. If $\tau_{i'} \geq \Delta$, then $F^i_j(v) = v_j$. As $j \notin \mathcal{J}(v)$, there exists $i'' \in I^*$ such that $i''j \in \mathcal{R}(v)$ and $v_j \geq \omega_j(\Gamma_{i''j})$. The symmetric version of Lemma 5.(iii), applied to $v \geq_J F^i(v)$, yields $i''j \in \mathcal{R}(F^i(v))$; hence, $j \notin \mathcal{J}(F^i(v))$. If $\tau_{i'} < \Delta$, then $ij \in \mathcal{R}(F^i(v))$ by (4) and $F^i_j(v) \geq \omega_j(\Gamma_{ij})$ by (10); again, we conclude that $j \notin \mathcal{J}(F^i(v))$. □

Proof of Theorem 2. Fix the order $(i_1, \ldots, i_m)$ used in F^I, and let $(v^s)_{s \geq 0}$ be the sequence of intermediate iterates starting at v: $v^0 = v$ and $v^{s+1} = F^{\kappa(s)}(v^s)$, where $\kappa(s) := i_q$ for $q - 1 \equiv s \pmod m$. Thus $(F^I)^{(z)}(v) = v^{mz}$. By Theorem 1, every v^s is stable, the sequence is weakly increasing with respect to $\geq_I$, and an application of F^i changes the payoff profile if and only if it strictly increases i's payoff.

Enumerate the steps $s_1 < s_2 < \ldots$ at which the targeted player's payoff strictly increases, and write

$$k_r := \kappa(s_r), \qquad y_r := v^{s_r+1}_{k_r}$$

for the corresponding player and her increased payoff. For any fixed player i, the values y_r with $k_r = i$ are strictly increasing. Every sweep that changes the payoff profile contains at least one of the steps s_r, so it suffices to bound the number of such steps uniformly.

Consider the following backward recursive construction starting at step s_r. Set $r_0 := r$. Given r_ℓ, apply the first formula in (5) at step s_{r_ℓ}: for some $i'_\ell \in I^*$,

$$y_{r_\ell} = \theta_{k_{r_\ell} i'_\ell} \qquad \text{or} \qquad y_{r_\ell} = v^{s_{r_\ell}}_{i'_\ell} + C_{k_{r_\ell} i'_\ell}.$$

If $y_{r_\ell} = \theta_{k_{r_\ell} i'_\ell}$, the construction stops. Otherwise, $y_{r_\ell} = v^{s_{r_\ell}}_{i'_\ell} + C_{k_{r_\ell} i'_\ell}$. If the payoff of i'_ℓ has not been strictly increased before step s_{r_ℓ}, then $y_{r_\ell} = v_{i'_\ell} + C_{k_{r_\ell} i'_\ell}$, and the construction also stops.

In the remaining case, let $s < s_{r_\ell}$ be the most recent step at which the payoff of i'_ℓ strictly increased. Since step s changes the payoff profile, Theorem 1.(ii) implies that the payoff of $\kappa(s)$ strictly increased at s, so $s = s_{r_{\ell+1}}$ for some $r_{\ell+1} < r_\ell$. If $i'_\ell \neq k_{r_{\ell+1}}$, the second formula in (5) implies that

$$v^{s_{r_{\ell+1}}+1}_{i'_\ell} = \max\big\{v^{s_{r_{\ell+1}}}_{i'_\ell},\ y_{r_{\ell+1}} - C_{k_{r_{\ell+1}} i'_\ell}\big\}.$$

Since the payoff of i'_ℓ strictly increases at step $s_{r_{\ell+1}}$, the second term attains the maximum, so $v^{s_{r_{\ell+1}}+1}_{i'_\ell} = y_{r_{\ell+1}} - C_{k_{r_{\ell+1}} i'_\ell}$. If $i'_\ell = k_{r_{\ell+1}}$, then $v^{s_{r_{\ell+1}}+1}_{i'_\ell} = y_{r_{\ell+1}}$ by the definition of $y_{r_{\ell+1}}$, and $C_{k_{r_{\ell+1}} i'_\ell} = C_{i'_\ell i'_\ell} = 0$ by Lemma 7.(i), so the same equation holds.

Since the payoff of i'_ℓ does not change between steps $s_{r_{\ell+1}}+1$ and s_{r_ℓ}, we obtain

$$y_{r_\ell} = v_{i'_\ell}^{s_{r_\ell}} + C_{k_{r_\ell} i'_\ell} = v_{i'_\ell}^{s_{r_{\ell+1}}+1} + C_{k_{r_\ell} i'_\ell} = y_{r_{\ell+1}} + C_{k_{r_\ell} i'_\ell k_{r_{\ell+1}}}, \tag{11}$$

where we write $C_{ii'i''} := C_{ii'} - C_{i''i'}$ for $i, i', i'' \in I^*$.

As the indices r_ℓ strictly decrease, the construction stops after finitely many applications of (11). Every y_r can thus be written as a terminal expression—either $\theta_{ii'}$ or $v_{i'} + C_{ii'}$—plus a sum of terms of the form $C_{ii'i''}$.

We next bound the number of applications of (11). Consider the sequence

$$k_{r_0}, k_{r_1}, k_{r_2}, \ldots$$

of players whose payoffs are strictly increased at the steps encountered in the backward construction, and split it into consecutive blocks of $m+1$ terms, discarding the remaining terms at the end. Since $|I^*| = m$, some term of a block coincides with an earlier term of that block. Let k_{r_q} be the first such term, and let k_{r_p} be the earlier term it repeats. Then $k_{r_p}, \ldots, k_{r_{q-1}}$ are distinct, so $q - p \le m$. Summing (11) from $\ell = p$ to $q-1$ yields

$$y_{r_p} - y_{r_q} = \sum_{\ell=p}^{q-1} C_{k_{r_\ell} i'_\ell k_{r_{\ell+1}}}.$$

The payoff of the player $k_{r_p} = k_{r_q}$ is strictly increased at both steps $s_{r_q} < s_{r_p}$. Therefore $y_{r_p} > y_{r_q}$, so the displayed difference is a positive element of the set X defined by

$$X := \Big\{ \sum_{\ell=0}^{q-1} C_{k_\ell i'_\ell k_{\ell+1}} \;\Big|\; 1 \le q \le m,\ k_0 = k_q,\ k_0, \ldots, k_{q-1} \text{ distinct},\ i'_0, \ldots, i'_{q-1} \in I^* \Big\}.$$

A block exists only if X contains a positive element. If so, let x denote the smallest positive element of X.

Over the steps $s_{r_q}+1, \ldots, s_{r_p}$, the sum of the payoffs of players in I^* increases by at least x, since

$$\sum_{i \in I^*} \big(v_i^{s_{r_p}+1} - v_i^{s_{r_q}+1} \big) \ge v_{k_{r_q}}^{s_{r_p}+1} - v_{k_{r_q}}^{s_{r_q}+1} = y_{r_p} - y_{r_q} \ge x.$$

The step ranges corresponding to distinct blocks are pairwise disjoint: a later block consists of terms with larger indices, which correspond to earlier steps, so its range ends at a step no later than s_{r_q}. As the sum of the payoffs of players in I^* is non-decreasing along the sequence (v^s), the increases over these ranges add up to at most the increase over the entire iteration. Since every intermediate payoff is stable, it lies between $\underline{v}^I$ and $\bar{v}^I$ under $\ge_I$, so the sum of the payoffs of players in I^* can increase by at most

$$A := \sum_{i \in I^*} (\bar{v}_i^I - \underline{v}_i^I)$$

over the entire iteration. Consequently, if the sequence contains b blocks, then $bx \le A$, and hence $b \le \lfloor A/x \rfloor$.

It follows that every backward construction involves at most B applications of (11), where

$$B := \begin{cases} (m+1)(\lfloor A/x \rfloor + 1), & \text{if } X \text{ contains a positive element} \\ m-1, & \text{otherwise.} \end{cases}$$

There are at most $2m^2$ possible terminal expressions—either $\theta_{ii'}$ or $v_{i'} + C_{ii'}$—and at most m^3 possible terms $C_{ii'i''}$ at each application of (11). Hence every y_r belongs to a set of at most

$$Q := 2m^2 \sum_{\ell=0}^{B} m^{3\ell}$$

values. For any fixed player i, the values y_r with $k_r = i$ are strictly increasing, so the payoff of i can be strictly increased at most Q times. There are therefore at most mQ steps s_r in total, and at most mQ sweeps change the payoff profile. Once a sweep leaves the payoff profile unchanged, the profile is a fixed point of F^I, and every later sweep leaves it unchanged as well. Hence, $(F^I)^{(z)}(v) = (F^I)^{(mQ)}(v)$ for all $z \geq mQ$, and the result holds with $Z := mQ$. □

Proof of Corollary 2. $\Phi^I = (F^I)^{(Z)}$ is a finite composition of the operators F^i for $i \in I^*$, each of which is isotone by Theorem 1. Hence, Φ^I is isotone.

To prove part (i), write $w := \Phi^I(v)$. Theorem 2 yields $F^I(w) = (F^I)^{(Z+1)}(v) = w$. Every F^{i_q} is weakly increasing with respect to $\geq_I$ by Theorem 1.(ii), and $F^{i_m} \circ \ldots \circ F^{i_1}(w) = w$, so every intermediate profile in the composition equals w. Therefore $F^{i_q}(w) = w$, and hence $i_q \notin \mathcal{I}(w)$, for every q, which means that $w \in \mathcal{S}^{**I}$.

For part (ii), $\Phi^I(v) \geq_I v$ follows from Theorem 1.(ii). If $\mathcal{I}(v) = \emptyset$, then $F^i(v) = v$ for all i, so $\Phi^I(v) = v$. If instead $\mathcal{I}(v) \neq \emptyset$, then $v \notin \mathcal{S}^{**I}$, while $\Phi^I(v) \in \mathcal{S}^{**I}$ by part (i), so $\Phi^I(v) \neq v$.

Part (iii) follows from mZ applications of Theorem 1.(iv). Part (iv) follows from parts (i) and (iii): if $v \in \mathcal{S}^{**J}$, then $\mathcal{J}(v) = \emptyset$, so $\mathcal{J}(\Phi^I(v)) = \emptyset$ and $\Phi^I(v) \in \mathcal{S}^{**I} \cap \mathcal{S}^{**J} = \mathcal{S}^{**}$. □

Proof of Lemma 9. Let $w := \bigwedge^I X \in \mathcal{S}$. For every $x \in X$, we have $x \geq_I w$, so Corollary 2.(ii) and isotonicity of Φ^I yield $x = \Phi^I(x) \geq_I \Phi^I(w)$. Hence $w \geq_I \Phi^I(w) \geq_I w$, so $w = \Phi^I(w) \in \mathcal{S}^{**I}$ by Corollary 2.(i). □

Proof of the second statement of Theorem 6. If $v \in \mathcal{S}^{**}$, then v is a fixed point of every F^i and F^j by Theorem 1.(ii) and its symmetric version, and hence also a fixed point of $F^J \circ F^I$.

Conversely, suppose that $F^J(F^I(v)) = v$, and write $w := F^I(v)$. If $i \in \mathcal{I}(v)$, then $F^i_i(v) = v_i + \Delta^i(v) > v_i$, and isotonicity of each $F^{i'}$ together with Theorem 1.(ii) yields $w_i \geq F^i_i(v) > v_i$. Thus $w_i > v_i = F^J_i(w)$, so i's payoff strictly decreases at some step F^j of the sweep defining $F^J(w)$. The symmetric versions of (4) and (10) make ji a residual pair after that step, with i's updated payoff at least $\omega_i(\Gamma_{ij})$, so i has no objection at that point. By the J-side counterpart of Theorem 1.(iv), she has no objection after the remaining steps either, so $i \notin \mathcal{I}(F^J(w)) = \mathcal{I}(v)$, a contradiction.

Therefore, $\mathcal{I}(v) = \emptyset$, implying that $w = v$ and $F^J(v) = v$. As in the proof of Corollary 2.(i), we conclude that $v \in \mathcal{S}^{**J}$. It follows that $v \in \mathcal{S}^{**}$. □

Proof of Lemma 10. Consider a demand round and a profile of demands v for which the maximum gap g is positive. Let G be the network formed by links between pairs that achieve the maximum gap g. Given our genericity assumption (which, as stated in footnote 6, is inherited in subgames), G cannot have cycles. Indeed, the existence of a cycle $(i_1, j_1, \ldots, i_{\bar{s}}, j_{\bar{s}})$ in G would imply that the matchings formed by the pairs $(i_1, j_1), \ldots, (i_{\bar{s}}, j_{\bar{s}})$ and $(i_2, j_1), \ldots, (i_1, j_{\bar{s}})$ generate the same total value. Since G has no cycles, there exists a node with a single link in G. Without loss of generality, assume that $j \in J$ has a single link, to some $i \in I$.

We first argue that i is matched at the first step of the round. The first step implements a maximum-cardinality matching among the links of G. If i were left unmatched at this step, then so would be j, whose only link in G is to i; the matching could then be augmented by the pair (i, j), contradicting maximum cardinality. Hence i is matched at the first step and receives the payoff $v_i + g/2$ in this round.

Let g' denote the greatest gap strictly smaller than g across all pairs of players present in the round (with $g' = -\infty$ if no such pair exists). Suppose player i deviates by raising her demand to $v_i + \varepsilon$, where $\varepsilon \in (0, \min(g, g - g'))$ is chosen so that $g - \varepsilon$ differs from the gap of every pair not involving i. Under the deviation, the gaps of all pairs involving i decrease by ε, while all other gaps remain unchanged. In particular, every pair involving i that achieved the gap g—the pair (i, j) among them—now has gap $g - \varepsilon$, and every other pair has gap at most $g' < g - \varepsilon$ or equal to g.

If every link of G involves i, then the greatest gap after the deviation is $g - \varepsilon > 0$, achieved only by pairs involving i, so i is matched at the first step at the gap $g - \varepsilon$. Otherwise, the first step implements a maximum-cardinality matching among the links of G not involving i, which leaves i and j unmatched; by maximum cardinality, no pair of players remaining after this step achieves the gap g. At the next step, the greatest gap among the remaining players is $g - \varepsilon$: the pair (i, j) achieves it, and every remaining pair not involving i has gap at most $g' < g - \varepsilon$. By the choice of ε, all pairs with gap $g - \varepsilon$ share the node i, so a maximum-cardinality matching of such pairs consists of a single pair involving i, and i is matched at this step.

In either case, i is matched in the same round at the gap $g - \varepsilon$, receiving the payoff $v_i + \varepsilon + (g - \varepsilon)/2 = v_i + g/2 + \varepsilon/2 > v_i + g/2$. The deviation is therefore profitable for i, a contradiction. □

Proof of Lemma 11. Consider a no-delay equilibrium in which player demands are given by a vector v, and a matching μ is implemented in the first demand round. Let $\tilde{v}$ denote the payoffs in this equilibrium. By Lemma 10, we have that $v_i + v_j \geq a_{ij}$ for all pairs $(i, j) \in I \times J$. Furthermore, whenever $\mu(i) = j$ for some $(i, j) \in I \times J$, it must be that $v_i + v_j = a_{ij}$, $\tilde{v}_i = v_i$ and $\tilde{v}_j = v_j$.

Let I' and J' denote the sets of players in I and J, respectively, matched under μ, and U' the set of players not matched under μ. The arguments above establish that

$$\tilde{v}_i + \tilde{v}_j \geq a_{ij}, \quad \forall i \in I', j \in J'. \tag{12}$$

By the definition of a no-delay equilibrium, we have that either $U' \subseteq I$ or $U' \subseteq J$, so $\tilde{v}_k = 0$ for $k \in U'$. Suppose that $U' \subseteq I$, and thus $I' = I \setminus U'$ and $J' = J$. If for some $i \in U'$ and $j \in J$, $v_j < a_{ij}$, then player i can demand $(a_{ij} - v_j)/2$, which guarantees that her match with j

has positive gap, and no match excluding i has a positive gap. Then, i would be matched "first" in the round and receive a positive payoff, a contradiction. It follows that if $U' \subseteq I$, then

$$\tilde{v}_i + \tilde{v}_j = v_j \geq a_{ij}, \quad \forall i \in U', j \in J. \tag{13}$$

Conditions (12) and (13) imply that $\tilde{v}$ is a stable payoff in this case. A symmetric argument leads to the same conclusion in the case $U' \subseteq J$.

By Lemma 1, we have that $\sum_{k\in N} \tilde{v}_k = \sum_{i\in I^*} a_{i\eta(i)}$. As $\sum_{k\in N} \tilde{v}_k = \sum_{i\in I:\mu(i)\neq\emptyset} a_{i\mu(i)}$, and η is the unique efficient matching, it must be that $\mu = \eta$. □

Proof of Theorem 8. For fixed (δ, β), consider a no-delay equilibrium in which first-round demands are given by v. The proof of Lemma 11 establishes that the equilibrium implements the efficient matching η, and yields stable payoffs $\tilde{v}$ given by $\tilde{v}_k = v_k$ for $k \in N^*$ and $\tilde{v}_k = 0$ for $k \in U$. By Lemma 10, the equilibrium demands v must satisfy $v_i + v_j \geq a_{ij}$. It follows that $v_i + v_{\eta(i)} = a_{i\eta(i)}$ for all $i \in I^*$.

Fix a player $i \in I^*$, and consider a deviation for this player to demanding $v_i' > v_i$ in the first round. Let $v' = (v_i', v_{-i})$ denote the new profile of demands, and μ denote the matching implemented in the first round of the demand phase under v'. We have that $v_i' + v_j > v_i + v_j \geq a_{ij}$ for all $j \in J$, so player i is not matched under μ.

To understand how the set of matched players differs under η and μ, construct the following chain of nodes. Let $i_0 = i$, $j_0 = \eta(i_0)$, and for $s \geq 0$, define recursively $i_{s+1} = \mu(j_s)$ and $j_{s+1} = \eta(i_{s+1})$, ending at the first step $\bar{s}$ where either (i) $\mu(j_{\bar{s}}) = \emptyset$ or (ii) $\eta(i_{\bar{s}+1}) = \emptyset$. The chain can never revisit a node, and thus must end in a finite number of steps (resulting in one of the two cases). Indeed, if the first instance of revisiting a node were $i_x = i_y \in I$ for $x < y$, then $x \geq 1$,[12] and $j_{x-1} = \mu(i_x) = \mu(i_y) = j_{y-1}$, contradicting the fact that i_x is the first revisited node. A similar contradiction obtains if the chain were to revisit a node in J first.

If the chain ends at step $\bar{s}$, all players in the set $\{i_0, j_0, \ldots, i_{\bar{s}}, j_{\bar{s}}\}$ are matched under η, and hence are not contained in U. Moreover, $ij_{\bar{s}} \in \mathcal{R}(\tilde{v})$. Indeed, for $0 \leq s \leq \bar{s} - 1$, we have that $\tilde{v}_{i_{s+1}} = v_{i_{s+1}} = v'_{i_{s+1}}$ and $\tilde{v}_{j_s} = v_{j_s} = v'_{j_s}$. Lemma 10 implies that $v_{i_{s+1}} + v_{j_s} \geq a_{i_{s+1}j_s}$, while $i_{s+1} = \mu(j_s)$ leads to $v'_{i_{s+1}} + v'_{j_s} \leq a_{i_{s+1}j_s}$. It follows that $\tilde{v}_{i_{s+1}} + \tilde{v}_{j_s} = a_{i_{s+1}j_s}$. Thus, $i_0, \ldots, i_{\bar{s}}$ is a rematching chain under $\tilde{v}$, and Lemma 3 implies that $ij_{\bar{s}} \in \mathcal{R}(\tilde{v})$.

Let us examine the two possibilities (i) and (ii) regarding the end of the chain. Suppose first that the chain ends in case (i) at a step $\bar{s}$ for which $\mu(j_{\bar{s}}) = \emptyset$. Then, i and $j_{\bar{s}}$ are matched under η, but not under μ. Consider the matching $\hat{\mu}$ that leaves only i_0 and j_0 unmatched and implements all other matches under η. For the profile of demands v', every pair matched under $\hat{\mu}$ achieves the maximum gap, and $\hat{\mu}$ contains $m - 1$ matches. Since μ leaves the opposite-side players i and $j_{\bar{s}}$ unmatched, it contains at most $m - 1$ matches. The maximum-cardinality requirement of the matching process therefore implies that μ contains exactly $m - 1$ matches. Since $\hat{\mu}$ matches $2(m-1)$ players in N^*, the tie-breaking rule in favor of players in N^* implies that μ also matches

[12] It is not possible that $x = 0$ because $i = i_0 = i_y = \mu(j_{y-1})$ leads to $\mu(i) = j_{y-1} \neq \emptyset$.

$2(m-1)$ players in N^*. As μ contains $m-1$ matches, every player matched under μ belongs to N^*: μ does not match any player in U, and i and $j_{\bar{s}}$ are the only players matched under η but not under μ.

We have thus established that the set of players not matched by μ—and continuing to play in the bargaining phase—is given by $\Gamma_{ij_{\bar{s}}} = \{i, j_{\bar{s}}\} \cup U$. For the deviation from v_i to v_i' not to be profitable for i, it must be that

$$v_i = \tilde{v}_i \geq \beta\delta\omega_i^\delta(\Gamma_{ij_{\bar{s}}}). \tag{14}$$

Suppose next that the chain ends in case (ii) at a step $\bar{s}$ for which $\eta(i_{\bar{s}+1}) = \emptyset$. This means that $j_{\bar{s}} \notin U$ and $i_{\bar{s}+1} \in U$. Since $i_{\bar{s}+1} = \mu(j_{\bar{s}})$, we have that $v'_{j_{\bar{s}}} \leq a_{i_{\bar{s}+1}j_{\bar{s}}}$, which in combination with $\tilde{v}_{j_{\bar{s}}} = v_{j_{\bar{s}}} = v'_{j_{\bar{s}}}$, leads to

$$\tilde{v}_{j_{\bar{s}}} \leq a_{i_{\bar{s}+1}j_{\bar{s}}}. \tag{15}$$

Consider now a sequence $(\delta, \beta) \to (1, 1)$ and an associated sequence of equilibria satisfying the two refinements in which players demand $v = v(\delta, \beta)$ in the first round, and the corresponding payoff profiles—denoted $\tilde{v}(\delta, \beta)$ to make the dependence on the frictions explicit—converge to a limit denoted by $\hat{v}$. Clearly, $\hat{v} \in \mathcal{S}$. We need to prove that $\mathcal{I}(\hat{v}) = \emptyset$ and $\mathcal{J}(\hat{v}) = \emptyset$.

Fix $i \in I^*$, and assume that v_i' in the arguments above is given by $v_i' = v_i + 1$. Since the set of possible μ is finite, the sequence (δ, β) admits a subsequence along which the corresponding μ is constant. It follows that the chain $\{i_0, j_0, \ldots, i_{\bar{s}}, j_{\bar{s}}\}$ associated with i's deviation from v_i to v_i' in the argument above is the same over this subsequence. In case (i), taking the limit in (14) over the subsequence, we obtain $\hat{v}_i \geq \omega_i(\Gamma_{ij_{\bar{s}}})$.

In case (ii), taking the limit in (15) over the subsequence, we obtain $\hat{v}_{j_{\bar{s}}} \leq a_{i_{\bar{s}+1}j_{\bar{s}}}$. As $i_{\bar{s}+1} \in U \cap I$, Lemma 4.(ii) rules out $\omega_i(\Gamma_{ij_{\bar{s}}}) > \hat{v}_i$, so $\hat{v}_i \geq \omega_i(\Gamma_{ij_{\bar{s}}})$.

The membership $ij_{\bar{s}} \in \mathcal{R}(\tilde{v}(\delta, \beta))$ holds at every point of the subsequence, and is supported by the same rematching chain. The binding stability constraints along the chain are preserved in the limit, so $ij_{\bar{s}} \in \mathcal{R}(\hat{v})$. In both cases, we conclude that $i \notin \mathcal{I}(\hat{v})$.

We have established that $\mathcal{I}(\hat{v}) = \emptyset$. An analogous argument shows that $\mathcal{J}(\hat{v}) = \emptyset$. □

Proof of Lemma 12. To prove the first statement, fix $v \in \mathcal{S}^{**}$ ($\mathcal{S}^{**} \neq \emptyset$ by Theorem 3). We show that v is a limit of elements of $\mathcal{S}^{**}$ that assign positive payoffs to all players in N^*. If $v_k > 0$ for all $k \in N^*$, there is nothing to prove. Otherwise, $v_k = 0$ for some $k \in N^*$. Without loss of generality, assume that $k \in I^*$, and use the notation $i := k$ going forward.

Since $v \in \mathcal{S}^{**}$, $i \in I^*$ and $v_i = 0$, there exists a rematching chain $i, \ldots, i'$ under v such that $\omega_i(\Gamma_{i\eta(i')}) = 0$. If i were the only element of I in the single-match game $\Gamma_{i\eta(i')}$, we would have $\omega_i(\Gamma_{i\eta(i')}) > 0$ (because all matches generate positive value). Hence, $U \neq \emptyset$ and $U \subset I$. If $v_j = 0$ for some $j \in J^*$, we would similarly conclude that $U \neq \emptyset$ and $U \subset J$, a contradiction. It follows that $v_j > 0$ for all $j \in J$.

Define

$$\begin{aligned}\mathcal{I}^0 &= \{i \in I^* \mid v_i = 0\}\\ \tilde{\mathcal{J}}^0 &= \bigcup_{i\in\mathcal{I}^0}\{j \in J^* \mid ij \in \mathcal{R}(v)\},\end{aligned}$$

and for $\varepsilon > 0$, construct a payoff profile w such that $w_{\eta(j)} = v_{\eta(j)} + \varepsilon$ and $w_j = v_j - \varepsilon$ for all $j \in \tilde{\mathcal{J}}^0$, and $w_k = v_k$ for all remaining players k.

For small enough $\varepsilon > 0$, arguments analogous to those in the proof of Theorem 1 show that $w \in \mathcal{S}$. First, note that for sufficiently small ε, we have $w_j \geq v_j - \varepsilon > 0$ for all $j \in J$. Then, the only remaining step in that proof that needs to be modified is the one verifying that no pair $(u, j) \in U \times J$ with $v_u + v_j = a_{uj}$ can block w. If $w_u + w_j < a_{uj}$ for such a pair, then $w_u = v_u = 0$ and $v_j \geq a_{uj}$ imply that $v_j = a_{uj}$ and $w_j = v_j - \varepsilon$. It follows that $j \in \tilde{\mathcal{J}}^0$, so there exists a rematching chain $i' = i_0, \ldots, i_{\bar{s}} = \eta(j)$ under v with $v_{i'} = 0$. However, this would imply there exists a matching μ that matches the pairs $(\eta(i_0), i_1), \ldots, (\eta(i_{\bar{s}-1}), i_{\bar{s}})$ as well as (u, j), and conforms with η for all remaining players different from i_0. Since $v_{i_0} = 0$, μ generates the same total value as η, contradicting our genericity assumption. Therefore, $w \in \mathcal{S}$.

If $j \in \tilde{\mathcal{J}}^0$, then $i'j \in \mathcal{R}(v)$ for some $i' \in \mathcal{I}^0$. Since v is stable and $v_{i'} = 0$, we have that $v_j \geq \max_{u\in U} a_{uj}$ and $v_j \geq a_{i'j}$. In any single-match game satisfying our genericity assumption, every player's OOP payoff is smaller than the maximum match value in that game. Thus, $v_j > \omega_j(\Gamma_{i'j})$. But $i'j \in \mathcal{R}(v)$ and $i' \in \mathcal{I}^0$ imply that $i'j \in \mathcal{R}(w)$ (this follows via Lemma 3, noting that any rematching chain under v with endpoints i' and $\eta(j)$ is preserved under the perturbed payoffs w). For $\varepsilon \leq v_j - \omega_j(\Gamma_{i'j})$, we have that $w_j = v_j - \varepsilon \geq \omega_j(\Gamma_{i'j})$, and hence $j \notin \mathcal{J}(w)$.

Consider next a $j \in J \setminus \tilde{\mathcal{J}}^0$. Then $v_j = w_j$. As $j \notin \mathcal{J}(v)$, there exists $i' \in I^*$ such that $i'j \in \mathcal{R}(v)$ and $v_j \geq \omega_j(\Gamma_{i'j})$, and the symmetric version of Lemma 5.(iii), applied to $v \geq_J w$, yields $i'j \in \mathcal{R}(w)$. Hence, $j \notin \mathcal{J}(w)$.

We conclude that $\mathcal{J}(w) = \emptyset$, so $w \in \mathcal{S}^{**J}$.

Now let $w' = \Phi^I(w)$. Corollary 2.(iv) implies that $w' \in \mathcal{S}^{**}$. By construction, $w_i > 0$ for all $i \in I^*$. Indeed, each $i \in \mathcal{I}^0$ has $i\eta(i) \in \mathcal{R}(v)$, so $\eta(i) \in \tilde{\mathcal{J}}^0$ and $w_i = \varepsilon$, while $w_i \geq v_i > 0$ for $i \in I^* \setminus \mathcal{I}^0$. Then, $w' \geq_I w$ implies $w'_i > 0$ for all $i \in I^*$. Moreover, applying the argument from the beginning of the proof to w', $U \subset I$ implies $w'_j > 0$ for all $j \in J^*$.

Hence $w'_k > 0$ for all $k \in N^*$. By (5), the mappings composed in Φ^I are continuous, and hence so is Φ^I. As $\varepsilon \to 0$, w converges to v, so $w' = \Phi^I(w)$ converges to $\Phi^I(v) = v$ by Corollary 2.(ii). This proves the first statement.

To prove the second statement, note that the hypothesis requires that for every $i \in I^*$ and $j \in J^*$, (i, j) is the unique value-maximizing match in Γ_{ij}. It follows that $\omega_i(\Gamma_{ij}), \omega_j(\Gamma_{ij}) > 0$. For every $v \in \mathcal{S}^{**}$ and $k \in N^*$, k does not have a legitimate objection to v, so there exists a residual pair $ij \in \mathcal{R}(v)$ involving k with $v_k \geq \omega_k(\Gamma_{ij})$. Then, $\omega_k(\Gamma_{ij}) > 0$ implies that $v_k > 0$. □

Proof of Theorem 9. Fix $\beta < 1$, and consider $v \in \mathcal{S}^{**}$ with $v_k > 0$ for all $k \in N^*$. For δ sufficiently close to 1, we construct an SPE in which players' demands in the first demand round are given by v, and the efficient matching η is implemented.

Suppose that a player $i \in I$ deviates from demanding v_i to demanding $v_i' > v_i$ in the first round. If $i \in U$, then the efficient matching is still implemented, and player i's payoff remains zero after the deviation.

Suppose that $i \notin U$. If it is possible to match m pairs of players (with zero gap), then $U \subset I$ and every such matching covers all of J^*. In this case, the tie-breaking rule in favor of players in N^* requires that a matching with this property maximizing the number of matched players in I^* be implemented.

Otherwise, the maximum number of possible matches does not exceed $m-1$. Then, $v \in \mathcal{S}^{**}$ implies that there exists a player $j \in J^*$ such that $ij \in \mathcal{R}(v)$ and $v_i \geq \omega_i(\Gamma_{ij})$. Lemma 3 yields a rematching chain $i_0 = i, i_1, \ldots, i_{\bar{s}} = \eta(j)$ under v. Following i's deviation, the market maker implements the matches $i_1\eta(i_0), \ldots, i_{\bar{s}}\eta(i_{\bar{s}-1})$. Furthermore, all remaining matches under η—$(i', \eta(i'))$ with $i' \in I \setminus (\{i_0, i_1, \ldots, i_{\bar{s}}\} \cup U)$—are implemented. The prescribed matching contains exactly $m-1$ matches. Since, in the case under consideration, no matching can contain m matches, it has maximum cardinality. Given that the only players in N^* it does not match are i and j, this matching is consistent with the tie-breaking rule in favor of players in N^*.

Given this matching process, after the first demand round, player i is left in a single-match game Γ_{ij} or a subgame in which only players from I are left without a match. Player i's continuation payoff in the bargaining game is $\beta\delta\omega_i^\delta(\Gamma_{ij})$ in the former case, and 0 in the latter. Since $v_i \geq \omega_i(\Gamma_{ij}) = \lim_{\delta\to 1}\omega_i^\delta(\Gamma_{ij})$, $v_i > 0$ and $\beta < 1$, for δ sufficiently close to 1, we have that $v_i > \beta\delta\omega_i^\delta(\Gamma_{ij})$. Hence, this is not a profitable deviation for i.

If $i \in I^*$ deviates to demanding $v_i' < v_i$, then i would be matched "first" with some $j \in J$ in the demand phase, receiving a payoff strictly smaller than $a_{ij} - v_j$. As $a_{ij} - v_j \leq v_i$ for all $j \in J$, this deviation cannot be profitable for i.

Deviations by players in J are handled symmetrically. These strategies are to be followed in every demand round as long as no match forms. We have verified that no player has an incentive to deviate from her first-round demand.

We are left to define behavior in other subgames that may arise in the demand phase following some matches. By the first statement in Lemma 12, in each such subgame, there exists a payoff profile in the corresponding refined core in which all players efficiently matched in the subgame get positive payoffs (this is possible because our genericity assumption regarding match values is preserved in every subgame—see footnote 6). We support this payoff profile with strategies analogous to those specified for the equilibrium path in the first round. Deviations by a single player from this behavior are handled in the same way. For all other profiles of demands submitted

in the first round of the subgame, the market maker selects a matching consistent with the rules of the game.[13]

Since the qualifiers "for δ sufficiently close to 1" above concern a finite set of subgames, all corresponding statements hold for δ above a uniform threshold $\underline{\delta} < 1$. The one-shot deviation principle then implies that the constructed strategies constitute a subgame perfect equilibrium for $\delta > \underline{\delta}$. It is easy to check that the constructed equilibrium satisfies both refinements. □

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